Network Effects, Market Structure and Industry Performance*

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Abstract

This paper provides a thorough analysis of oligopolistic markets with positive demand-side network externalities and perfect compatibility. The minimal structure imposed on the model primitives is such that industry output increases in a firm's rivals' total output as well as in the expected network size. This leads to a generalized equilibrium existence treatment that includes guarantees for a nontrivial equilibrium, and some insight into possible multiplicity of equilibria.

We formalize the concept of industry viability and show that it is always enhanced by having more firms in the market. We also characterize the effects of market structure on industry performance, with an emphasis on departures from standard markets. As per-firm profits need not be monotonic in the number of competitors, we revisit the concept of free entry equilibrium for network industries. The approach relies on lattice-theoretic methods, which allow for a unified treatment of various general results in the literature on network goods. Several illustrative examples with closed-form solutions are also provided.

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1 Introduction

It has often been observed that the nature of competition is qualitatively different in network industries. The presence of interdependencies in consumers' purchasing decisions induces demand-side economies of scale that highly affect market behavior and performance. When such effects prevail, be they of the snob or bandwagon type, purchase decisions are strongly influenced by buyers' expectations, leading to behavior not encompassed by traditional demand theory (Veblen, 1899 and Leibenstein, 1950).

From an industrial organization perspective, these distinctive features raise new questions and impose some challenges from a methodological perspective. In their pioneering work on markets with network effects, Katz and Shapiro (1985) developed the concept of fulfilled expectations Cournot equilibrium, which was widely adopted. The resulting literature on the topic has established a number of results that distinguish network markets from ordinary markets.[1]

The purpose of the present paper is to provide a thorough theoretical investigation of markets with homogeneous goods and network externalities, which unifies and extends the existing studies and tackles a number of new issues of interest that were either not previously addressed or only partially studied. We consider oligopolistic competition amongst firms in a market characterized by positive (direct) network effects when the products of the firms are perfectly compatible with each other, so the relevant network is industry-wide. While the current literature is more concerned with the case of firm-specific networks, three arguments justify our choice. First, several important industries fit the perfect compatibility framework, in particular those in the telecommunications sector, such as fax machines and phones, but also many classical industries such as fashion, automobiles, entertainment, etc...[2] Second, there are still several outstanding issues, which, although addressed in the growing literature on network externalities, have not been fully articulated from a modeling perspective, and thus remain less than fully understood from a theoretical standpoint. Third, a good understanding of the single network case can shed quite some light on the incentives for compatibility faced by firms in the case of firm-specific networks.

In its unifying scope, with an emphasis on minimal and economically meaningful assumptions on the market primitives, the paper provides a general existence result for non-trivial equilibria (i.e.

[1] See Economides and Himmelberg (1995), Economides (1996), Shy (2001), and Kwon (2007), among others.

[2] For some of these industries, each customer may have in mind his own social network only, as opposed to the overall industry network, when making a purchase decision, but we follow much of the literature in industrial organization in ignoring this distinction.

those with positive production), a uniqueness argument, and an extensive inquiry into the effects of market structure (or exogenous entry) on market performance. In terms of novel questions, the paper offers a general treatment of the critical issue of industry start-up, including the role of the number of firms in the market; some insight into the notion that the presence of expectations can substantially broaden the scope of possible outcomes relative to standard Cournot oligopoly; and a new look at the notion of free entry equilibrium into network industries. Throughout, the paper takes a comparative perspective in that new findings are contrasted with their standard Cournot counterparts, in an attempt to shed light on the novel features of network industries.

The underlying approach is to impart minimal monotonicity structure to the oligopoly model at hand, which achieves the twin goals of ensuring the existence of a fulfilled expectations Cournot equilibrium while at the same time allowing clear-cut predictions on the comparative statics of market performance with respect to the number of firms. The critical structure is imposed on the model in the form of two economically meaningful complementarity conditions on the primitives that guarantee the key properties that, along a given firm's best response, industry output increases in rivals' total output as well as in the expected network size. The overall analysis relies on lattice-theoretic methods.[3] This approach allows us to unify in a common setting the existing results in the literature on network goods, considerably weakening the required conditions, as well as to derive important new results. A key benefit of the approach is to allow for more transparent economic intuition behind the cause-effect relationships we analyze.

We next provide a more detailed overview of our findings, coupled with a literature review. The problem of existence of fulfilled expectations Cournot equilibrium proceeds in two distinct steps. To establish abstract existence via Tarski's fixed point theorem, we adopt the arguments of Amir and Lambson (2000) and Kwon (2007) that directly exploit the monotonicity structure discussed above. However, as expectations about the size of the network is a key determinant of consumers' willingness to pay in these industries, the trivial, no production, equilibrium is often part of the equilibrium set. When this is the case, our previous proof of existence is not of much interest; it uses powerful methods to establish existence, but the underlying equilibrium may a priori be the trivial one, the presence of which can be characterized in more direct fashion. As a consequence we complete the analysis by offering a second set of (stronger) conditions that ensure the existence of (at least) one non-trivial equilibrium, i.e. one with strictly positive sales.

Although our model is static in nature, we construct an explicit dynamics mapping consumers'

[3] See Topkis (1978), Vives (1990), Milgrom and Roberts (1990) and Milgrom and Shannon (1994).

expectation of the network size to the corresponding Cournot industry output to analyze the viability of the industry. This tatonnement-type dynamics is quite natural and has tacitly been the basis of many discussions of the viability issue in the literature. Some studies that consider telecommunications markets, such as Rohlfs (1974) and Economides and Himmelberg (1995), suggest that network industries typically have three equilibria. Under this natural dynamics, the two extreme equilibria are stable in expectations and the middle equilibrium (usually called critical mass) is unstable. The argument behind this structure is quite simple for pure network goods: If consumers expect that few buyers will acquire the good, then the good will be of little value to consumers and few will end up buying it. These low sales in turn further depress consumers' expectations through the above dynamics, and the market unravels towards the trivial (or no-trade) equilibrium. However, if expectations are higher to start with, other, non-trivial, equilibria will also be possible. This argument is often used to explain the start-up problem in network industries, or the difficulties faced by the incumbent firms in attempting to generate enough expectations to achieve critical mass.

An important aim of the present paper is to shed light on the role of market structure as a determinant of the viability of a network industry, a novel issue that, somewhat surprisingly, has not yet been addressed in the literature. We find that the presence of more firms in the market always enhances industry viability.

Regarding market performance, the extremal equilibria (i.e. maximal and minimal) lead to an industry output that increases in the number of firms, n, as in standard Cournot competition. On the other hand, as this also implies an increase in the equilibrium network size or expectations, the output result does not imply that market price decreases in n. Thus, the so-called property of quasi-competitiveness, which under similar assumptions holds in standard Cournot competition, does not hold here.[4] In addition, when n increases per-firm equilibrium output increases if the demand is not too log-concave and decreases otherwise.

The most drastic departure from standard oligopoly lies in the effects of entry on per-firm profits. Whenever per-firm outputs and the market price increase (decrease) with n, per-firm profits increase (decrease) in n as well.[5] The conclusion that competition may increase each firm's profit is quite provocative and leads to several important implications, both from theoretical and policy-oriented

[4] A Cournot market is said to be quasi-competitive if the equilibrium market price decreases with the number of firms in the industry.

[5] This result already appears in the context of a model with an inverse demand function that is linear in output and no costs of production in Economides (1996), who in turn formalizes a remark made by Katz and Shapiro (1985).

perspectives. We explore in some detail the consequences on the concept of free entry. We show that free entry is quite indeterminate in industries with network externalities, and propose the concept of strong free entry equilibrium as a refinement that leads to the free entry equilibrium with the largest number of firms as unique outcome. However, this refinement requires some pre-play communication amongst firms without the possibility of making binding agreements. Such coordination, though pro-competitive in that it increases competition, may well engender antitrust action. In addition, since the incumbent firms in the market may prefer to see further entry by new firms, a number of policy issues may need a fresh look and some revisiting. There may be more scope for pro-competitive cooperation or coordination by firms. One might observe a higher propensity for licensing, possibly coupled with lower royalty rates; less patenting or a permissive attitude towards patent infringement; as well as more product standardization in industries where each firm might possess its own separate network of consumers. These likely policy consequences are similar to those one might expect to see as a consequence of the result that having more firms alleviates the start-up problem for the industry. In short, when more competition can be necessary to get the industry started up, or to enhance each firm's profit in an ongoing industry, the usual trade-offs between consumer surplus and producer surplus are no longer the norm, and it is not surprising that many pillars of conventional wisdom about market behavior and appropriate antitrust policy might need revisiting. In particular, new avenues for cooperation amongst firms that compete in the product market might open up in network industries. Proper reaction to these new incentives for coordinated action by market competitors might well require a substantial overhaul of existing antitrust policy (Shapiro, 1996). This in turn ought to rely on extensive theoretical analysis focusing on the special nature of industries with network externalities, and this is a primary motivation of the present work.

The effects of entry on industry performance as reflected in social welfare, consumer surplus and industry profits also display some distinctive features as compared to standard Cournot competition. The demand-side economies of scale weaken the conditions under which social welfare and industry profits increase with more entry. Alternatively, if the cross-effect on the inverse demand function is positive, it is possible that consumer surplus decreases with n. Katz and Shapiro (1985) explain, in a similar (although not identical) situation, the intuition behind this result: If the network externality is strong for the marginal consumer, then the increase in the expected network caused by the change in the number of firms will raise his or her willingness to pay for the good by more than that of the average consumer. As a consequence, the firms will be able to raise price by more than the increase in the average consumer's willingness to pay for the product and consumer surplus will fall.

Another noteworthy aspect of this paper is that we provide several explicit examples with easy closed-form solutions to illustrate in a simple way some of the conclusions we derive. In particular, Example 1 captures most of the relevant features often associated with the telecommunication industry in the literature.[6]

The paper is organized as follows. Section 2 presents the model, introduces the equilibrium concept and the main assumptions. Section 3 proves existence of trivial and non-trivial equilibria, and provides conditions for the equilibrium to be unique. Section 4 discusses the scope for network effects to broaden the set of possible outcomes. Section 5 studies industry viability. Section 6 analyzes output, price and per-firm profits as a function of the number of firms in the market. Section 7 deals with free entry equilibrium in markets with network effects. Section 8 looks at market performance as reflected in social welfare, consumer surplus and aggregate profits, again, as a function of n. Section 9 contains all the proofs of this paper. Finally, an elementary and self-contained review of the lattice-theoretic notions and results needed here forms the Appendix.

2 The analytical framework

This section presents the standard oligopoly model with network effects along with the commonly used equilibrium concept due to Katz and Shapiro (1985). In view of the more general nature of our treatment, we enumerate all the needed assumptions we shall use later and their justification.

We consider a static model to analyze oligopolistic competition in industries with positive network effects, reflected in consumers' willingness to pay being increasing in the number of other agents acquiring the same good. We assume the firms' products are homogeneous and perfectly compatible with each other, so there is a single network comprising the outputs of all firms in the industry.

The market is fully described by the inverse demand function $P(Z,S)$ and the number of identical firms n, each having cost function $C(x)$, where x denotes the firm's output, Z aggregate output in the market and S the expected size of the network. The cost of producing no output is zero. Considering that each consumer buys at most one unit of the good, S also stands for the expected number of people buying the good. Sometimes, it will be useful to express the production

[6] Some of the examples we construct below do not satisfy all the assumptions in this paper. As the violations are not critical in any way and analytical examples (with nice closed-form solutions that capture the features we want to highlight) are hard to come by, we are not concerned by this issue.

side in terms of average cost $A(x)$, defined as $C(x)/x$ with $A(0) = C'(0)$.

For a given S, each firm's reaction correspondence is obtained by maximizing the profit function $\pi(x, y, S) = xP(x+y, S) - C(x)$

$$\widetilde{x}(y, S) = \arg\max\{\pi(x, y, S) : 0 \leq x \leq K\} \tag{1}$$

where x is the firm's level of output, y the output of the other $(n-1)$ firms in the market and $K > 0$ the production capacity of each firm.

At equilibrium, all relevant quantities x, y, Z and π will be indexed by the underlying number of firms n, e.g., we shall denote Z_n the equilibrium industry output corresponding to n firms in the market, and x_{in} the equilibrium output of firm i. When clear from the context, we will avoid the subindex i in the latter variable.

Each firm chooses its output level to maximize its profits under the assumptions that (i) consumers' expectations about the size of the network, S, is given; and (ii) the output level of the other firms, y, is fixed. Alternatively, we may think of the firm as choosing total output $Z = x + y$, given the other firm's cumulative output, y, and the expected size of the network, S, in which case, with $\widetilde{\pi}(Z, y, S) = (Z - y)P(Z, S) - C(Z - y)$

$$\widetilde{Z}(y, S) = \arg\max\{\widetilde{\pi}(Z, y, S) : y \leq Z \leq y + K\}. \tag{2}$$

Consistency requires $\widetilde{Z}(y, S) = \widetilde{x}(y, S) + y$.

An equilibrium in this game is a vector $(x_{1n}, x_{2n}, ..., x_{nn})$ that satisfies the following conditions

1. $x_{in} \in \arg\max\left\{xP\left(x + \sum_{j\neq i} x_{jn}, S\right) - C(x) : 0 \leq x \leq K\right\}$; and
2. $S = \sum_i x_{in}$.

Since the seminal paper by Katz and Shapiro (1985), this notion of equilibrium, known as "Fulfilled Expectations Cournot Equilibrium (FECE)", has become standard for oligopolies with network effects. It requires that both consumers and firms correctly predict the market outcome, so that their beliefs are confirmed in equilibrium, i.e., expectations are rational. While strategic in their choice of outputs in the usual Cournot sense, firms are "network-size taking" in their perceived inability to directly influence customers' expectations of market size. One plausible justification for this is that firms are unable to credibly commit to output levels that customers could observe and reliably use in formulating expectations about network size (Katz and Shapiro, 1985).[7]

[7] Were such commitment credible for firms, standard Cournot equilibrium with inverse demand function $P(Z, Z)$

Viewing S as an inverse demand shift variable, condition 1 just describes the equilibrium in standard Cournot competition with exogenous S. Let $z_n(S)$ denote the corresponding industry output equilibrium correspondence. Adding condition 2, an aggregate output $Z_n \in z_n(S)$ constitutes a FECE industry output if it satisfies $Z_n = S$ as well. As a consequence, if we graph $z_n(S)$ as a function of S, the FECE industry outputs are all the points where this correspondence crosses the 45° line. This idea will play a key role in both the proof of existence and the viability analysis.

Another, fully game-theoretic, interpretation of this equilibrium notion is in the context of a two-stage game, wherein a market maker (or a regulator) announces an expected network size S in the first stage, and firms compete in Cournot fashion facing inverse demand $P(Z,S)$ in the second stage. If the market maker's objective function is to minimize $|S - z_n(S)|$, then to any subgame-perfect equilibrium of this game corresponds a FECE of the Cournot market with network externalities, and vice-versa.

Whenever well-defined, we denote the maximal and minimal points of a set by an upper and a lower bar, respectively. Thus, for instance, $\overline{Z}_n$ and $\underline{Z}_n$ are the highest and lowest industry equilibrium outputs when there are n firms in the market.

Denote by $W(Z,S) \triangleq \int_0^Z P(t,S)\,dt - ZA(Z/n)$ the Marshallian social welfare when aggregate output is Z, all firms produce the same quantity and the expected size of the network is S. Similarly, consumer surplus is $CS(Z,S) \triangleq \int_0^Z P(t,S)\,dt - ZP(Z,S)$.

We now list the assumptions used in this paper, starting with a set of standard ones, followed by more substantive assumptions.

The standard assumptions are

(A1) $P(.,.)$ is twice continuously differentiable, $P_1(Z,S) < 0$ and $P_2(Z,S) > 0$.

(A2) $C(.)$ is twice continuously differentiable and increasing.

(A3) $x_i \leq K$, for all firm i.

These are all commonly used assumptions, including $P_2(Z,S) > 0$, which reflects positive network effects, or the property that consumers' willingness to pay increases in the expected number of people who will buy the good. Assumption A3 imposes capacity constraints in the production process of each firm, a convenient assumption to force compact output sets in a setting where firms

would be the more appropriate concept. A direct comparison between these two concepts also appears in Katz and Shapiro (1985).

may otherwise wish to produce unbounded output levels. Our results do not rely in any way on K taking on any particular sets of values, as in Amir and Lambson (2000).

The second set of assumptions are placed on two functions that play a key role in the overall analysis. Let $\Delta_1(Z,y)$ denote the cross-partial derivative of $\widetilde{\pi}(Z,y,S)$ with respect to Z and y, and $\Delta_2(Z,S)$ the cross-partial derivative of $\log P(Z,S)$ with respect to Z and S, scaled by $[P(Z,S)]^2$,

$$\begin{aligned}\Delta_1(Z,y) &= -P_1(Z,S) + C''(Z-y) \text{ and}\\ \Delta_2(Z,S) &= P(Z,S)\,P_{12}(Z,S) - P_1(Z,S)\,P_2(Z,S)\end{aligned}$$

The domains of Δ_1 and Δ_2 are $\varphi_1 \equiv \{(Z,y) : y \geq 0, Z \geq y\}$ and $\varphi_2 \equiv \{(Z,S) : Z \geq y, S \geq 0\}$ respectively, both of which are lattices (in the product order).

The second set of assumptions is

(A4) $\Delta_1(Z,y) = -P_1(Z,S) + C''(Z-y) > 0$ on φ_1.

(A5) $\Delta_2(Z,S) = P(Z,S)\,P_{12}(Z,S) - P_1(Z,S)\,P_2(Z,S) > 0$ on φ_2.

(A6) $P(Z,S)\,P_{11}(Z,S) - [P_1(Z,S)]^2 < 0$ on φ_2.

Assumptions A4 and A5 guarantee that the profit function $\widetilde{\pi}(Z,y,S)$ has strictly increasing differences on φ_1 and the strict single-crossing property in $(Z;S)$, respectively. A4 allows for limited scale economies in production, and has been justified in detail by Amir and Lambson (2000). A5 has the precise economic interpretation that the elasticity of demand increases in the expected network size S.[8] In his pioneering study of the elementary microeconomic foundations of interdependent demands, Leibenstein (1950) suggested that demand is more elastic in network markets because individual reactions to price changes are followed by additional reactions, in the same direction, to each other's change in consumption.[9] A5 essentially captures the cumulative effect of these mutually reinforcing effects on aggregate demand. Another plausible interpretation of A5 is that it formalizes the concept of demand-side scale economies that is often postulated as a characteristic of network effects in the literature, though not in a precise manner. In terms of the model structure, the direct effects of A4 and A5 in the upcoming analysis are that $\widetilde{Z}(y,S)$ increases in y and S, respectively.

[8] The price elasticity of demand is $-\left(\frac{\partial P(Z,S)}{\partial Z}\frac{Z}{P(Z,S)}\right)^{-1} = -\left(Z\frac{\partial \log P(Z,S)}{\partial Z}\right)^{-1}$, which is increasing in S if and only if $\log P(Z,S)$ has increasing differences in (Z,S) (Topkis, 1998, p. 66).

[9] Although Leibenstein referred to the concept of positively interdependent demands as "bandwagon effect", it is essentially identical to the network effect we analyze in this paper.

A6 holds that $P(Z,S)$ is log-concave in Z. This is a generalized concavity condition that guarantees that $\widetilde{Z}(y,S)$ is a single-valued function. As most results in this paper do not require the latter property, A6 is crucially needed only for the uniqueness result, Theorem 7.

3 Existence and uniqueness of equilibrium

In this section we provide a general abstract equilibrium existence result, exploiting the minimal monotonic structure of the model reflected in A4-A5. Then we derive additional sufficient conditions that guarantee the existence of a non-trivial equilibrium, i.e. one with strictly positive industry output. We finally provide conditions for the equilibrium to be unique.

3.1 Existence of FECE

We begin with the central monotonicity result, which is a direct consequence of A4 and A5.

Lemma 1 *Assume A1-A5 are satisfied. Then, every selection of the best-response correspondence $\widetilde{Z}(y,S)$ is increasing in both y and S.*

This lemma leads to an abstract existence result for symmetric equilibrium, along with the fact that the same assumptions preclude the possibility of asymmetric equilibria.

Theorem 2 *Assume A1-A5 are satisfied. Then, for each $n \in N$, the Cournot oligopoly with network effects has (at least) one symmetric equilibrium and no asymmetric equilibria.*

Theorem 2 extends the existence results in the literature of network goods to a very general setting, dispensing with both the assumptions of no cross-effects on the demand side and constant marginal costs of production that are commonly found in the literature (Katz and Shapiro, 1985, Economides and Himmelberg, 1995, and Economides, 1996, among others).

Comparing the assumptions we impose here with those in standard Cournot competition, the only new requirement is that the price elasticity of demand increases with the network size (A5), taking $P_2(Z,S) > 0$ as a natural property of network markets. Analogs of all other assumptions are also needed for proving existence in the standard Cournot model, as reflected in Theorem 2.1, Amir and Lambson (2000), as seen next.

Lemma 3 *Assume A1-A4 are satisfied. Then, for any given $n \in N$,* *(i) the standard Cournot oligopoly (with exogenous S) has a symmetric equilibrium and no asymmetric equilibria;*

(ii) if in addition A5 holds, the maximal and minimal selections of $z_n(S)$, $\overline{z}_n(S)$ and $\underline{z}_n(S)$, increase in S; and

(iii) if in addition A5-A6 hold, $z_n(.)$ is a single-valued and continuous function.

In network markets, the trivial (zero-production) outcome, $(Z, S) = (0, 0)$, is often an equilibrium. This phenomenon intensifies when the network good has little stand-alone value (i.e., $P(Z, 0)$ is small). The telecommunication industries, such as faxes, phones and e-mails, typically exhibit this characteristic. Given any of these goods, if end users believe no one else will acquire it, the good will have no value, and the trivial outcome will necessarily be part of the equilibrium set.

In such markets, Theorem 2 is not of much interest since the underlying equilibrium may a priori be the trivial one, the presence of which can be characterized in more direct fashion. To complete the picture, we first provide necessary and sufficient conditions for the trivial equilibrium, and then add some extra assumptions to ensure the existence of (at least) one non-trivial equilibrium.

Lemma 4 *For any $n \in N$, the trivial outcome is an equilibrium if and only if*

$$xP(x, 0) \leq C(x) \text{ for all } x \in [0, K]. \tag{3}$$

Thus if the trivial outcome is an equilibrium for some n, it remains an equilibrium for all $n \in N$.

This lemma simply says that the trivial equilibrium arises if and only if when the common expectation (amongst firms and consumers) about the size of the network is zero, and a firm believes the other firms will produce no output, the best it can do under (3) is to produce zero as well. The proof follows directly from the definition of FECE. It also states that if the trivial equilibrium prevails for a given n, the industry will admit the trivial equilibrium for any other number of firms.

Building on Theorem 2, the next result provides alternative sufficient conditions to ensure the existence of a non-trivial equilibrium, i.e. one with strictly positive industry output.

Theorem 5 *Assume A1-A5 are satisfied. There exists a non-trivial equilibrium if at least one of the following conditions is also fulfilled*

(i) zero is not an equilibrium output (i.e. (3) does not hold);

(ii) zero is an equilibrium output, $P(0,0) = C'(0)$*,* $P_1(0,0) + P_2(0,0) > 0$ *and*

$n > [-P_1(0,0) + C''(0)] / [P_1(0,0) + P_2(0,0)]$*; or*

(iii) zero is an equilibrium output, $C''(.) \geq 0$ *and for some* $S \in (0, nK]$*, some* $\widehat{Z} \geq S$ *and all* $Z \leq S$*,*

$$(n-1)\int_{Z}^{\widehat{Z}} P(t,S)\,dt + \left[\widehat{Z}P\left(\widehat{Z},S\right) - ZP(Z,S)\right] - n^2\left[C\left(\widehat{Z}/n\right) - C(Z/n)\right] \geq 0. \tag{4}$$

In Theorem 5 (i), a non-trivial equilibrium exists as a consequence of Theorem 2, as it guarantees the existence of at least one equilibrium. Then, if the trivial one is not part of the equilibrium set, there must be an equilibrium with a strict positive industry output. This result captures the case most often investigated in the literature, dealing with a high stand-alone value for the network good.[10]

Under the conditions in Part (ii), although $z_n(0) = 0$, $z_n(S)$ starts above the 45° line near 0, implying the existence of a non-trivial equilibrium in view of Theorem 2. Formally, this follows from applying Tarski's Theorem to $z_n(S)$ for $S \in [\epsilon, nK]$, given some $\epsilon > 0$ small enough. As expected, the stronger the network effect around the origin is, as captured by $P_2(0,0)$, the less stringent the existence condition for the non-trivial equilibrium gets (i.e. the lower the threshold value of n is).

Condition (4) ensures that, although $z_n(0) = 0$, $z_n(S)$ is above the 45° line at some $S \in (0, nK]$, so a non-trivial equilibrium exists by Tarski's Theorem applied to $z_n(.)$ mapping $[S, nK]$ to itself. An interpretation of (4), involving the evaluation of a weighted combination of welfare and profits (see Bergstrom and Varian, 1985), is given in the Appendix (see Lemma 25).

The proof of Theorem 5 uses the following intermediate result, which also plays a key role in the viability analysis (Section 4).

Lemma 6 *Assume A1-A5 are satisfied. If* $0 \in z_n(0)$*, then* $z_n(0) = 0$*, i.e.* $z_n(0)$ *is single-valued. If in addition* $P(0,0) = C'(0)$*, the slope of* $z_n(.)$ *is also single-valued and right-continuous at* 0*, and*

$$z_n'(0) = \frac{nP_2(0,0)}{-(n+1)P_1(0,0) + C''(0)}. \tag{5}$$

If the trivial equilibrium is not interior, i.e. $P(0,0) < C'(0)$*, then* $z_n'(0) = 0$*.*

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[10] Such goods include high-tech products such as computer systems, software and various home electronic appliances, but also some classical goods and services such as movies, restaurants, night clubs, fashion goods, etc...

This lemma shows the trivial equilibrium has interesting properties. Although $z_n(.)$ might a priori be multi-valued, i.e. a correspondence, when zero is part of the equilibrium set, it is single-valued at the origin. If in addition the trivial equilibrium is interior, the slope of this function is given by (5) and depends on n.

3.2 Uniqueness of FECE

Theorems 2 and 5 prove existence, but they do not eliminate the possibility of multiple equilibria, which, in markets with network effects, constitutes more of a norm than an exception. Multiple equilibria are due to the positive feedback typical of network industries that derives from expectations. If consumers believe the good will not succeed, it will usually fail. On the contrary, if they expect it to succeed, it will usually succeed.

In this subsection, we assume A6 is satisfied, in addition to A1-A5. The added benefit of A6 is to ensure that $z_n(.)$ is single-valued and continuous, as shown in Lemma 3 (iii). Although A6 is sufficient for uniqueness in standard Cournot competition (see Amir and Lambson, 2000), the same result here requires an additional condition related to the function

$$f(Z,n) \triangleq \frac{n\left\{\left[P(Z,Z)-C'(Z/n)\right]P_{12}(Z,Z)-P_1(Z,Z)P_2(Z,Z)\right\}}{(n+1)\left[P_1(Z,Z)\right]^2-n\left[P(Z,Z)-C'(Z/n)\right]P_{11}(Z,Z)-P_1(Z,Z)C''(Z/n)}. \tag{6}$$

The function $f(Z,n)$ describes the slope of $z_n(S)$ with respect to S along the diagonal path, i.e., at $S=Z$ (see Proofs). The next theorem shows uniqueness of FECE.

Theorem 7 *In addition to A1-A6, assume $f(Z,n)<1$ for all Z in $[0,nK]$. Then, the Cournot game with network effects has a unique and symmetric FECE, which coincides with the trivial equilibrium if and only if (3) holds.*

The proof of this theorem is simple. Assuming A1-A6 are satisfied, $z_n(.)$ is single-valued and continuous. If in addition $f(Z,n)$ is everywhere lower than one, then the slope of $z_n(.)$ along the diagonal is lower than one as well. The uniqueness result now follows directly from this observation, since any two adjacent fixed points of $z_n(.)$ must include one for which $f(Z,n)$ is larger than one.

The assumption in Theorem 7 is a sufficient, but not necessary, condition to ensure uniqueness, in the tradition of methods based on degree theory (Dierker, 1972). Although it is not globally satisfied in the less general model of Katz and Shapiro (1985), their equilibrium is unique anyway.

4 On the theoretical scope of network effects

In view of the need for an expectations-based equilibrium concept instead of one of the standard concepts of oligopolistic behavior centered on Nash equilibrium, it is natural to investigate the extent to which the presence of these expectations enlarges the scope of possible outcomes in network industries. One meaningful way to frame such a question is to characterize the class of functions that could emerge as possible equilibrium industry outputs given network size S, i.e., as possible selections of the Cournot equilibrium correspondence $z_n(S)$.[11] Some simple insights into this question can be derived by considering, with a given number of firms n, zero production costs and the specific inverse demand (see Amir and Lambson, 2000), with $h' \geq 0$,

$$P(Z, S) = \exp[-nZ/h(S)]. \tag{7}$$

For regular Cournot oligopoly (with demand function (7) and exogenous S), there is a *unique Cournot equilibrium* and it is in dominant strategies: $x^* = h(S)/n$, so $Z^* = h(S)$. Hence firms have constant reactions curves, and may thus be viewed as essentially non-strategic and fully predictable in their behavior.

The FECE solve the fixed-point relation $Z = h(S) = S$. Since f is so far an arbitrary function, $h(S) = S$ may have no solutions at all (if h does not intersect the 45° line), or as many solutions (or FECE points) as h has fixed points. In particular, if h is taken to be the identity function, anything at all is a FECE, and the model has no predictive power whatsoever!

This argument shows rather strikingly the scope of possible new outcomes that expectations or network effects can generate, which have no counterparts in the corresponding regular Cournot oligopoly. Indeed, this illustration has an "anything goes" flavor of a rather extreme kind. This construction also illustrates the potential for multiple equilibria in the presence of network effects, along with new issues to face for testing such models (Echenique and Komunjer, 2009).

This construction will be invoked repeatedly below to design nice closed-form examples that illustrate particular results.

[11] This is reminiscent of the question of what constitutes a valid aggregate excess demand function in general equilibrium theory, which led to the well-known Debreu-Mantel-Sonnenschein theorem (Mas-Colell, Whinson and Green, 1995). Likewise, it is also in the same vein as the issue of what constitutes a valid optimal policy in Ramsey-type dynamic optimization models (Boldrin and Montrucchio, 1986). The answers provided in these two different settings were similarly broad, in that any function with minimal regularity conditions is a valid outcome function. Hence, the conclusions were that the two underlying theories impose very little structure on their respective outcome functions.

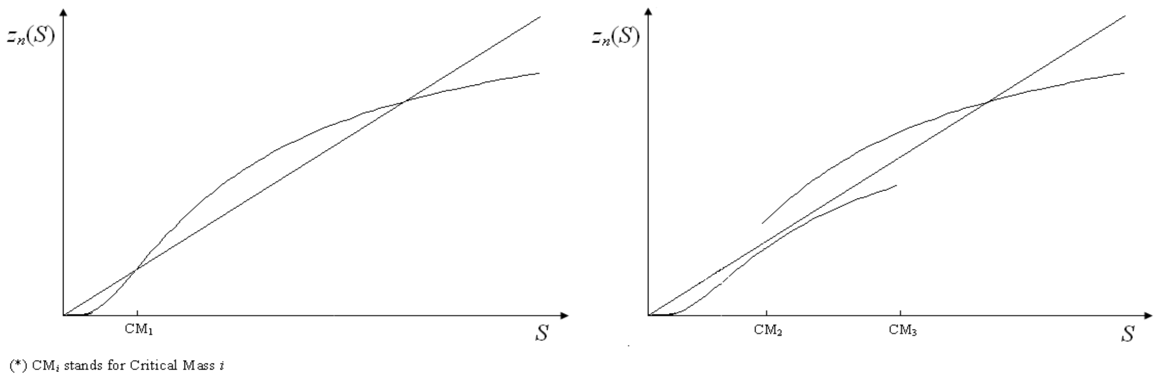

Figure 1: Viability and Basin of Attraction of the Trivial Equilibrium

5 Industry viability

Many studies suggest that the left panel of Figure 1 reflects the structure of the telecommunication industries. The underlying game there displays three possible equilibria, the trivial equilibrium, a middle unstable equilibrium, usually called critical mass, and a high stable equilibrium.[12] The justification of this configuration is quite simple: If all the consumers expect that no one will acquire the good, then the good has no value and no one will end up buying it, resulting in the trivial equilibrium for the industry. However, if expectations are higher to start with, another, non-trivial, equilibria will prevail.

Whenever the trivial equilibrium is locally stable in expectations (as in Figure 1), one possibility is that the market never emerges as a result of an expected size of the network that is too low to start with. In view of the equilibrium concept adopted here, the incumbent firms are simply unable to influence these expectations to get them past the critical mass. Under such conditions, even if the industry does get going, Cournot equilibrium on the basis of low expectations cannot lead firms to produce enough output to generate expectations beyond the critical mass, and the industry will unravel through a natural process towards the trivial equilibrium. This argument is commonly invoked to capture the start-up problem that frequently affects these markets, and is often referred to as the "chicken and egg" paradox. Oren and Smith (1981) offer an early discussion of this phenomenon in electronic communication markets.

[12] There are several definitions of the notion of critical mass in the literature, some in dynamic settings and others in static settings. In the present paper, we wish to adapt the most common definition, which is as the smallest non-zero (Cournot-) unstable FECE, to our framework taking into account the multi-valuedness of $z_n(S)$. The formal definition is given below.

The tacit dynamic process underlying this analysis can be formalized through the following expectations/network size recursion, starting from any initial $S_0 \geq 0$, where $\widehat{z}_n$ will denote either the maximal or minimal selection, but sometimes any increasing selection, of z_n, as will be specified,

$$S_k = \widehat{z}_n \left(S_{k-1}\right), k \geq 1. \tag{8}$$

This process thus begins with a historically given initial expectation S_0, then postulates that firms react by engaging in Cournot competition with demand $P(Z, S_0)$, leading to an industry output $\widehat{z}_n \left(S_0\right)$. The latter will in turn determine consumers expectation $S_1 \in \widehat{z}_n \left(S_0\right)$, and the process repeats indefinitely. This yields a sequential adjustment process in which consumers and firms behave myopically with respect to the size of the network. Taking a single-valued selection of $z_n(S)$ amounts to selecting one particular Cournot equilibrium for each given S.

For each increasing selection of $z_n(S)$, denoted $\widehat{z}_n \left(S\right)$, we can formally define the corresponding critical mass as the smallest initial expectation $\widehat{S}_0$ such that for all $S_0 > \widehat{S}_0$, the orbit given by (8) converges to a nonzero FECE. This definition captures the notion of critical mass irrespective of whether the selection at hand is continuous, or continuous from one side only (i.e. right or left), or neither, at the critical mass (see below). In the right panel of Figure 1, there is a whole interval of critical masses, each corresponding to a different monotonic selection of $z_n(S)$.

As we shall consider explicitly the dynamics in (8) as given by the two extremal selections, as reflected in the right side of Figure 1, the usual notions of stability need to be adapted accordingly.[13]

Definition 8 *The trivial equilibrium is best-case (worst-case) stable if there is a right neighborhood V of 0 such that for all S_0 in V, the orbit $S_k = \overline{z}_n \left(S_{k-1}\right) \to 0$ $(S_k = \underline{z}_n \left(S_{k-1}\right) \to 0)$, as $k \to \infty$.*

Here, the qualification of best-case and worst-case refers to the type of Cournot equilibrium selection given network size S. Indeed, as is intuitive, the maximal (minimal) selection is most (least) favorable for the viability of the industry. Note that in the right panel of Figure 1, CM_2 and CM_3 are the best-case and worst-case critical masses, respectively.

Let $V_n^b(V_n^w)$ denote the largest set of values of S_0 for which the trivial equilibrium is best-case (worst-case) stable. We shall refer to V_n^b and V_n^w as the best and worst-case basins of attraction of the trivial equilibrium, respectively.

In view of Lemma 6, both $\overline{z}_n$ and $\underline{z}_n$ are continuously differentiable at 0 with $\overline{z}_n' \left(0\right) = \underline{z}_n' \left(0\right)$. Assuming henceforth that this derivative is (generically) not equal to 1, 0 is an isolated equilibrium

[13] Related issues are addressed in some detail in Echenique (2002).

(for a formal proof, see e.g., Granas and Dugundji, 2003, p. 326-327). Since in addition, $\overline{z}_n$ and $\underline{z}_n$ are increasing in S, both V_n^b and V_n^w are intervals.[14] In the left panel of Figure 1, as $z_n(.)$ is single-valued, these two intervals coincide and are equal to $(0, CM_1)$; in the right panel of this figure $\overline{z}_n$ and $\underline{z}_n$ induce $V_n^b = (0, CM_2)$ and $V_n^w = (0, CM_3]$, respectively.[15]

Each industry can be classified into one of three possible categories in terms of best-case or worst–case viability.

Definition 9 *An industry is said to be*

(i) best-case (worst-case) uniformly viable if every orbit in (8) with $\widehat{z}_n = \overline{z}_n(\underline{z}_n)$ converges to some non-zero equilibrium starting from any $S_0 > 0$;

(ii) best-case (worst-case) conditionally viable if, for $\overline{z}_n(\underline{z}_n)$, the same convergence as in (i) takes place only from sufficiently high S_0; and

(iii) best-case (worst-case) nonviable if every orbit in (8) with $\widehat{z}_n = \overline{z}_n(\underline{z}_n)$ converges to 0 from any $S_0 \geq 0$.

This definition extends in the obvious way to any increasing selection of $z_n(S)$, in which case one simply removes the qualifiers "best-case" and "worst-case".

Thus, for any increasing selection $\widehat{z}_n(S)$, the critical mass is 0 if the industry is uniformly viable, ∞ if it is nonviable, and satisfies if the industry is conditionally viable:[16] $\widehat{S}_0 > 0$ and

$$\lim_{S \uparrow \widehat{S}_0} \widehat{z}_n(S) \leq \widehat{S}_0, \widehat{z}_n(S_0) \leq \lim_{S \downarrow \widehat{S}_0} \widehat{z}_n(S)$$

The next result provides sufficient conditions for each viability outcome by linking it to our previous results on the existence of a non-trivial equilibrium.

Proposition 10 *If A1-A5 are satisfied, an industry is*

(i) worst-case uniformly viable if and only if either condition (i) or (ii) of Theorem 5 holds;

(ii) best-case conditionally viable if condition (iii) of Theorem 5 holds; and

[14] Since z_n is u.h.c., $\underline{z}_n = \min z_n$ is l.s.c. and left-continuous, and $\overline{z}_n = \max z_n$ is u.s.c. and right-continuous. Hence, V_n^b is open at its upper bound while V_n^w may or may not be.

[15] The fact that V_n^b is open and V_n^w is right-closed follows from both Figure 1 and Footnote 14.

[16] In other words, there will always exist a unique critical mass for a given selection. A formal proof can easily be given, relying only on the well-known properties of monotonic functions, in particular that all discontinuities are of the first kind.

(iii) best-case nonviable if the conditions of Lemma 4 and Theorem 7 hold.

To provide a basis for comparing two different situations that might prevail for the same industry, we need to formalize a partial order for increasing viability.

Definition 11 *The best-case (worse-case) viability of an industry is said to increase if either*

(i) the industry goes from best-case (worst-case) nonviable to best-case (worst-case) conditionally viable, or from the latter to best-case (worst-case) uniformly viable; or

(ii) the industry is best-case (worst-case) conditionally viable and $V_n^b(V_n^w)$ contracts.

The next result shows that additional firms in the market can only enhance the viability of the network industry.[17] Examples 1 and 2, at the end of this section, illustrate this key effect.

Theorem 12 *Assume A1-A5 are satisfied. Then,*

(i) both $\overline{z}_n(.)$ and $\underline{z}_n(.)$ shift up as n increases. Hence, having more firms in an industry always increases its best-case and worst-case viability; and

(ii) if the trivial outcome is an equilibrium (i.e. (3) holds) and $P_1(0,0) + P_2(0,0) \leq 0$, an industry cannot be uniformly viable for any n (even in best-case).

Theorem 12 captures the key role of market structure on industry viability: Having more firms around implies that a lower critical mass would be needed to launch a given industry. The underlying intuition is intimately connected to the FECE concept, as discussed next. Consider the natural question: In case S_0 happens to be below critical mass, why can't the existing firms attempt to act as if there were more of them by producing a higher output level in an effort to influence consumers' expectations of the network size upwards? In a context where the appropriate solution concept is FECE, firms presumably cannot commit to their desired output levels in a credible way, and, likewise, attempting to inflate their number by committing to a higher output would also not be credible, and would thus not constitute behavior compatible with the FECE concept.

This result is consistent with observed market behavior. The fax market took decades beyond the discovery of the technology to get started (Shapiro and Varian, 1998). Now and then, an attempt

[17] Economides and Himmelberg (1995) show that, under specific conditions, the market structure has no effect on the critical mass. Although Theorem 12 seems to contradict their finding, our results do not coincide because we define critical mass in a different way.

at launching a new product with network effects is seen to fail. One plausible diagnosis according to the present analysis is an insufficient number of firms at the early stages of the emerging industry.

In industries with multiple firms having their own versions of the same general good, this result might explain why firms often settle for full compatibility between their products, instead of incompatibility. Their objective is to generate a single industry network that would be viable, when separate networks with one firm each would not be. This implies that some form of cooperation amongst direct rivals could be needed for their products to succeed. One example is the case of Sony and Philips, who jointly created industry standards for compact disc in the mid 80's (Shapiro, 1996). Such forms of cooperation have no counterparts in non-network industries.

Example 1. Consider the symmetric Cournot oligopoly with no production costs, and inverse demand function given by

$$P(Z,S) = \exp\left(-\frac{2Z}{\exp(1-1/S)}\right) \text{ with } Z, S \in [0, nK].$$

The reaction function of a firm is $\widetilde{x}(y,S) = (1/2)\exp(1-1/S)$. Since each firm has a dominant strategy, $\widetilde{x}(y,S)$ does not depend on y, and we can add the reaction functions to obtain

$$z_n(S) = (n/2)\exp(1-1/S).$$

An equilibrium industry output solves $z_n(Z) = Z$ in Z. Then we have: $Z_1 = \{0\}$, $Z_2 = \{0,1\}$, $Z_3 = \{0, 0.457, 2.882\}$ and $Z_4 = \{0, 0.373, 4.311\}$, as shown in Figure 2.

As can be easily seen, the trivial equilibrium is always stable. With only one firm in the market, this is the only equilibrium so the industry is nonviable. With one extra firm, a larger equilibrium appears and the industry becomes conditionally viable (barely so as $z_n(.)$ is tangent to the 45° line). For a larger number of firms, the equilibrium configuration encompasses three equilibria; the two extreme are stable and the intermediate one is unstable. This last equilibrium, often called critical mass, decreases in n. This is an exact closed-form example of the three-equilibrium constellation that is often portrayed as typical in many network industries.

Here, $z_n(.)$ shifts up as n increases (cf. Theorem 12). The industry goes from nonviable to conditionally viable as n goes from 1 to 2 firms. As n further increases, viability increases since the basin of attraction of 0 shrinks, but uniform viability is never attained since $P_1(0,0) + P_2(0,0) = 0$ (cf. Theorem 12). □

In our first example, initial expectations must be high enough to start the market up (when $n \geq 2$). Although the critical mass shrinks as the number of firms increases, the start-up problem

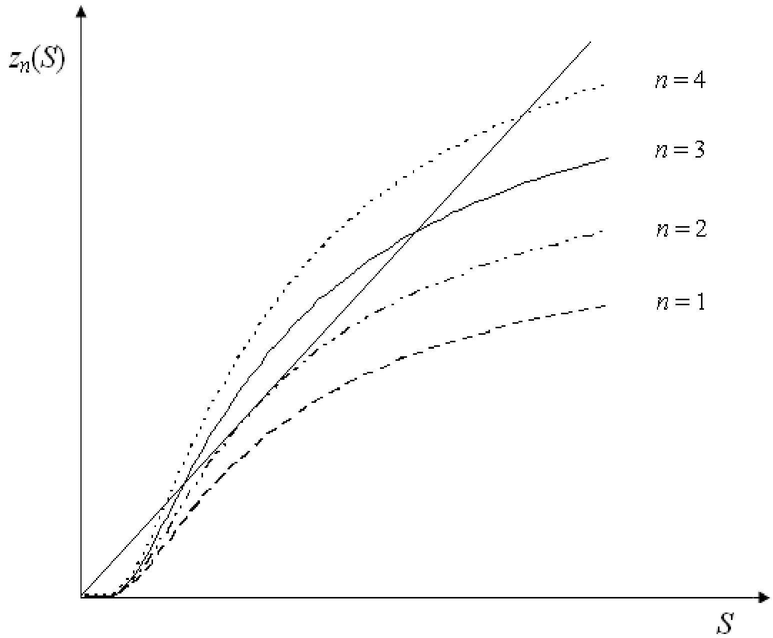

Figure 2: Viability and Market Structure

always prevails. The next example shows an extreme case where this problem disappears when the number of firms is sufficiently large.

Example 2. Let us consider an alternative inverse demand function

$$P(Z,S) = e^{-\frac{Z}{bS}} \text{ with } Z,S \in [0,nK] \text{ and } b > 0.$$

The reaction function of any given firm is

$$\widetilde{x}(y,S) = \begin{cases} bS & \text{if } S < K/b \\ K & \text{if } S \geq K/b \end{cases}.$$

Then the equilibrium correspondence of the standard Cournot with exogenous S is

$$z_n(S) = \begin{cases} nbS & \text{if } S < K/b \\ nK & \text{if } S \geq K/b \end{cases}.$$

As equilibrium industry output solves $z_n(Z) = Z$, we have three possible equilibrium configurations

$$Z_n = 0 \text{ if } n < 1/b \qquad Z_n \in [0,nK] \text{ if } n = 1/b \qquad Z_n = \{0,nK\} \text{ if } n > 1/b.$$

Figure 3 illustrates our results for $b = 1/2$, assuming there are 1, 2 and 3 firms in the market.

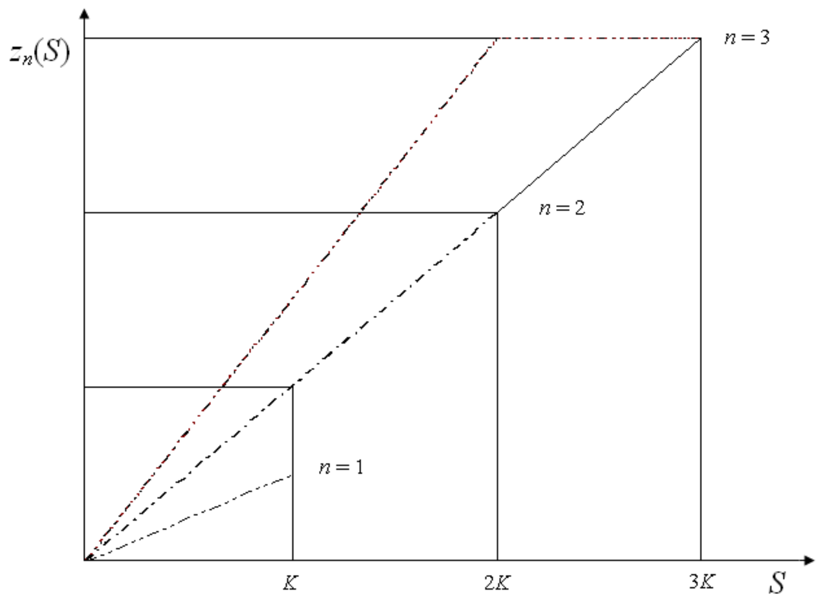

Figure 3: Viability and Market Structure

We observe in the graph that when $n = 1$ the slope of $z_n(S)$ with respect to S is everywhere less than one, and the industry is nonviable. When $n = 2$ the industry becomes uniformly viable (barely so, as $z_n(.)$ coincides with the 45° line). Finally, for all $n > 2$ every orbit in (8) converges to nK starting from any $S_0 > 0$, so the industry is uniformly viable as well. □

The previous examples illustrate two different situations where the presence of network effects might have unusual implications on firms' attitudes towards intellectual property rights and entry deterrence. Indeed, firms will not be tempted to engage in entry deterrence activities if their number is insufficient to start-up an industry. In such a case, those in possession of patents will have a much higher incentive than usual to engage in licensing, and will even find it in their interest to give away their patents to their competitors. Naturally, such generosity will prevail only until the industry is started up, or until profits cease to increase with the number of competitors, as we shall see below.

6 Equilibrium price, outputs and profits

This section studies the effects of market structure on the equilibrium industry output, per-firm output, market price and per-firm profits. Amir and Lambson (2000) address similar questions regarding the standard Cournot competition, our results show that network effects introduce new interesting features.

The next analysis makes all the statements on the largest equilibrium, i.e. the one with the largest equilibrium outputs, namely, $\overline{Z}_n$ and $\overline{x}_n$. When the trivial outcome is an equilibrium, it is also the smallest equilibrium. Since it is invariant in the number of firms, the comparative statics questions below are of no interest for that equilibrium. When the trivial outcome is not an equilibrium, then our conclusions also apply to the minimal selections, $\underline{Z}_n$ and $\underline{x}_n$.

Our first theorem relates new entry to equilibrium industry output and market price.

Theorem 13 *Assume conditions A1-A5 are satisfied. Then, we have*

(i) the extremal equilibrium industry outputs, $\overline{Z}_n$ *and* $\underline{Z}_n$*, increase in* n; *and*

(ii) if $P_1(Z,Z) + P_2(Z,Z) \geq (\leq)\, 0$ *on* $I_n = [\overline{Z}_n, \overline{Z}_{n+1}]$*, then* $P\left(\overline{Z}_{n+1}, \overline{Z}_{n+1}\right) \geq (\leq)\, P\left(\overline{Z}_n, \overline{Z}_n\right).$

Theorem 13 (i) is also true in standard Cournot competition, as shown by Amir and Lambson (2000), Theorem 2.2 (b). In the latter, the usual law of demand suffices for the market price to decrease after new entry. As Part (ii) indicates, the effect of entry on market price is ambiguous when network effects prevail. The reason is that when industry output increases the firms must set the price low enough to attract the marginal consumer, but when more buyers joint the network consumers' willingness to pay increases. Thus the overall effect of entry on the market price depends on how relevant the output effect is as compared to the network effect. As a consequence, the so-called property of quasi-competitiveness, which under similar assumptions holds in the standard Cournot game, is not satisfied here.

To make inferences about the effects of entry on equilibrium per-firm outputs and profits, we need to introduce a new function

$$g(Z) = \left[P(Z,Z) - C'(Z/n)\right]\left[P_{11}(Z,Z) + P_{12}(Z,Z)\right] - P_1(Z,Z)\left[P_1(Z,Z) + P_2(Z,Z)\right]. \qquad (9)$$

Theorem 14 *In addition to A1-A5, assume* $\overline{Z}_n$ *and* $\overline{Z}_{n+1}$ *are interior equilibria and let* $I_n = [\overline{Z}_n, \overline{Z}_{n+1}]$*. Then, we have*

(i) if $g(Z) \geq 0$ *on* I_n*, the largest per-firm equilibrium output increases in* n*, i.e.* $\overline{x}_{n+1} \geq \overline{x}_n$; *and*

(ii) if $g(Z) \leq 0$ *on* I_n*, the largest per-firm equilibrium output decreases in* n*, i.e.* $\overline{x}_{n+1} \leq \overline{x}_n$.

In short, this result holds that the scope for the business-stealing effect, which is nearly universal in standard Cournot oligopoly (at least in a global sense), is quite a bit narrower in the presence of network externalities. On the other hand, the scope for the opposite, or business-enhancing, effect is much broader in the present setting, as we see next.

Corollary 15 *In addition to the conditions of Theorem 14, assume no costs of production. Then* $\overline{x}_{n+1} \geq \overline{x}_n$ *if*

$$[P(Z,Z) P_{12}(Z,Z) - P_1(Z,Z) P_2(Z,Z)] + \left[P(Z,Z) P_{11}(Z,Z) - P_1^2(Z,Z)\right] \geq 0 \qquad (10)$$

on I_n*, for which log-convexity of* $P(Z,S)$ *in* Z *is a sufficient condition.*

The left-hand side of (10) is the same as $g(Z)$ when the firms face no production costs. Its first term is positive by A5, and log-convexity of $P(Z,S)$ in Z ensures the second one is positive as well. Therefore log-convexity is a sufficient, but not necessary, condition for the highest per-firm equilibrium output to increase after new entry. Amir and Lambson (2000), Theorem 2.3, require log-convexity to globally ensure the same result for standard Cournot competition. Hence network effects facilitate this unusual outcome.

Based on Theorems 13 and 14, the following result deals with the effects of entry on per-firm equilibrium profits. Recall that in standard Cournot oligopoly, the only part of the conventional wisdom about the effects of competition that is universally valid is that per-firm profits decline with the number of competitors (Amir and Lambson, 2000, and Amir, 2003). We now see that in the presence of network effects, this result can easily be reversed.

Theorem 16 *In addition to A1-A5, assume* $\overline{Z}_n$ *and* $\overline{Z}_{n+1}$ *are interior equilibria and let* $I_n = [\overline{Z}_n, \overline{Z}_{n+1}]$*. Then, we have*

(i) if $P_1(Z,Z) + P_2(Z,Z) \geq 0$ *and* $g(Z) \geq 0$ *on* I_n*, at the largest equilibrium,* $\pi_{n+1} \geq \pi_n$*; and*

(ii) if $P_1(Z,Z) + P_2(Z,Z) \leq 0$ *and* $g(Z) \leq 0$ *on* I_n*, at the largest equilibrium,* $\pi_{n+1} \leq \pi_n$*.*

The first result provides sufficient conditions for the firms in the market to prefer further entry by new firms. It generalizes a result in Economides (1996), based on a more specific formulation, which in turn formalizes a remark made by Katz and Shapiro (1985).

Although surprising, the intuition for this outcome is simple. New entry increases the equilibrium industry output, as shown in Theorem 13, and a direct effect is that market price goes down by the usual law of demand. But via the effect on the size of the network, this output increase also shifts the inverse demand function up, thus pushing for a price increase. Then if the overall effect on the market price is positive and each firm increases own output, the existing firms in the market are better-off after new entry. As Economides (1996) states, if the externalities are strong, the network effect dominates the usual competitive effect of entry.

A natural question arises when profits increase in n. Why can't the existing firms attempt to act as if there were more of them in order to each reap higher profits at equilibrium? Since they would do so by producing a higher output level in an effort to influence consumers' expectations of the network size upward, the answer is the same as for the start-up problem: The tacit lack of commitment power on the part of the firms, which is at the heart of the FECE concept.

Corollary 17 *In addition to the conditions of Theorems 14 and 16, assume* $P_{11}(Z,Z)+P_{12}(Z,Z)=0$*, for all* Z*. If* $P_1(Z,Z)+P_2(Z,Z)\geq(\leq)\,0$ *on* I_n*, then, at the largest equilibrium,*

(i) per-firm equilibrium output increases (decreases) in n*, i.e.* $\overline{x}_{n+1}\geq(\leq)\,\overline{x}_n$*; and*

(ii) per-firm equilibrium profits increase (decrease) in n*, i.e.* $\pi_{n+1}\geq(\leq)\,\pi_n$*.*

The new condition in Corollary 17, $P_{11}(Z,Z)+P_{12}(Z,Z)=0$, is satisfied if, for example, $P(Z,S)=h(S)-kZ$ with $h(.)$ an increasing function, or $P(Z,S)=f(S-Z)$ with $f(.)$ increasing on the reals.

We end this section with an example that highlights the implications of Theorem 16, and then explain how these results affect the standard characterization of the free entry number of firms.

Example 3. Consider the symmetric Cournot oligopoly with no production costs, and inverse demand function given by

$$P(Z,S)=\max\{a+bS^{\alpha}-Z,0\},\ \text{with } Z,S\in[0,nK],\ a\geq 0,\ b>0 \text{ and } \alpha\in(0,1).$$

Assuming K is large enough, the reaction function of any given firm is

$$\widetilde{x}(y,S)=\max\{(a+bS^{\alpha}-y)/2,0\}.$$

After a simple computation, the symmetric equilibrium industry output is implicitly defined by

$$-Z_n(1+n)+na+nbZ_n^{\alpha}=0.$$

Let $a=10$, $b=5$ and $\alpha=4/5$. Using a numerical approach, per-firm equilibrium profits for different values of n are

$$\begin{aligned}\pi_1 &\approx 14,561<\pi_2\approx 49,255<\pi_3\approx 67,316<\pi_4\approx 70,676\\ \pi_5 &\approx 67,288>\pi_6\approx 61,520>\pi_7\approx 55,301>\pi_8\approx 49,404>...>\pi_{21}\approx 14,444.\end{aligned}$$

We observe that when the number of firms is small, $n = 1, 2$ or 3, the incumbent firms will be better off if an extra firm enters the market. When $n \geq 4$, firms will be worse-off after new entry.

Consider for instance a situation where entry costs are $14,440$, say. Then a single firm would barely make a positive profit, and potential entrants might decide to stay out if they based their assessment on standard oligopoly settings (due to profit just covering entry costs). Yet, the market should actually accommodate a full 21 firms at the unique free entry equilibrium! □

The next section explores some other consequences of the presence of network effects on the well-known concept of free entry equilibrium.

7 Free entry and FECE

Consider the standard problem of free entry as a two-stage game (e.g. Mankiw and Whinston, 1986). In the first stage, each of an infinite number of firms decides whether to enter the industry or not, knowing the entry cost EC. In the second stage, upon observing the number of entrants, firms engage in standard Cournot competition. The free entry (subgame-perfect) equilibrium number of firms n^e is then defined by

$$\pi_{n^e} \geq EC \text{ and } \pi_{n^e+1} < EC. \tag{11}$$

These conditions simply state that the n^e firms that entered and those that did not do not regret their decisions. Assuming a unique Cournot equilibrium in the second stage, the free entry equilibrium number of firms is uniquely defined (ignoring the integer constraint) by the zero-profit condition $\pi_{n^e} = EC$ since π_n is always decreasing in n.

In the present setting with network effects, we can also define free entry equilibrium as a subgame-perfect equilibrium of the two-stage game, upon replacing the Cournot equilibrium in the second stage by a FECE selection, assumed to be a non-trivial one below.[18] We now investigate the consequences of keeping the standard definition of free entry equilibrium as given in (11).

In light of Theorem 16, the concept of free entry equilibrium may not be as well-behaved for network industries.[19] The equilibrium number of firms n^e need not be uniquely defined as π_n may intersect the horizontal line at EC more than once, with the free entry equilibria being only those for which this intersection is from above to below.[20]

[18] As an abuse of terminology, we ignore here the fact that FECE is not a fully game-theoretic concept.

[19] To begin with, FECE are often not unique, so to each FECE corresponds at least one free entry equilibrium. In particular, the presence of the trivial equilibrium would lead to no entry always being a free entry equilibrium.

[20] In Example 3, it is of interest to observe that, although π_n is inverse U-shaped in n, the free entry number of

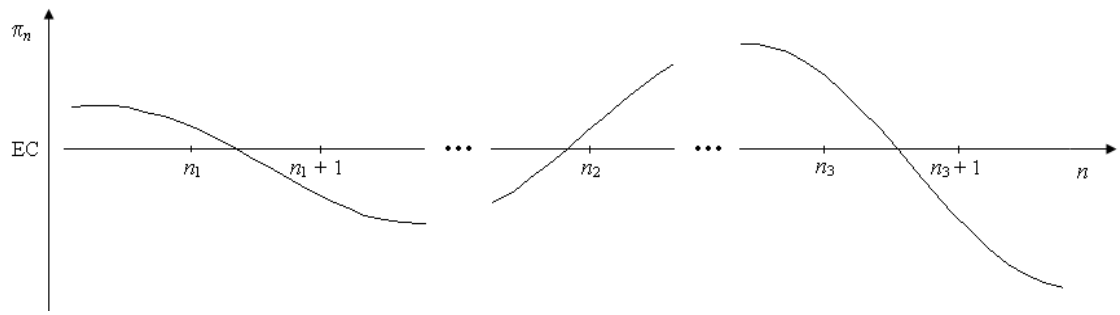

Figure 4: Free-entry and Network Effects

To fix ideas, let us focus on the situation depicted in Figure 4. There are two free entry equilibria, n_1 and n_3 according to the definition given above. We assume in what follows that $\pi_{n_3} > EC$ and $\pi_{n_3+1} < EC$, which holds generically (thus, in contrast to much of the literature, we are not ignoring the integer constraint, and this will turn out to be crucial below). Clearly, with n_1 firms in the market, no single firm outside the market would wish to deviate and enter on its own. However, a group of $n_3 - n_1$ firms outside the market could stage a coalitional deviation – all enter the market – that would be beneficial to all coalition members. With n_3 firms in the market, no coalition has a profitable deviation (we are assuming that, consistent with Figure 4, $\pi_n < EC$ for all $n > n_3$).[21] Hence, there is a unique strong Nash equilibrium, which is also coalition-proof, in the one-shot game with payoffs written as functions of the two possible first-stage actions (enter and do not enter) for each firm, given a non-trivial FECE in the second stage. This induces what we might refer to as a unique strong (and coalition-proof) free entry equilibrium with n_3 firms in. The latter is also the Pareto-dominant free entry equilibrium if $\pi_{n_3} \geq \pi_{n_1}$, but not otherwise.[22]

Clearly, the underlying ideas behind this discussion are quite general. In cases where there are multiple free entry equilibria, only one is also coalition-proof and it is always the free entry equilibrium with the highest number of firms in the market. This equilibrium clearly has a lot of

firms is nevertheless uniquely defined for any level of EC.

[21] Here again, we are using the assumption $\pi_{n_3} > EC$, since without it, there would be a profitable coalitonal deviation to the n_1 equilibrium in case $\pi_{n1} > \pi_{n_3}$.

[22] A general remark about the common simplifying assumption of ignoring the integer constraint in the standard oligopoly literature is in order. If $\pi_{n^e} = EC$ holds, then each firm is indifferent between entering and not entering. So if one of the n^e firms went out of the market, we would still have a free entry equilibrium, which actually Pareto-dominates the original n^e-firm equilibrium since $\pi_{n^e-1} > \pi_{n^e}$. Thus ignoring the integer constraint is not as innocuous as it seems. One way out is to assume the (rather arbitrary) tie-breaking rule that, when indifferent, a firm always chooses to enter.

intuitive appeal from an applied perspective, since the free entry number of firms is often thought of as the largest number of firms that a market can sustain. It is thus reasonable to suggest coalition-proofness as a refinement to the notion of free entry equilibrium in markets with network effects.[23]

On the other hand, for this equilibrium to obtain, some pre-play communication without the option of making binding agreements might well be needed, as is the case with coalition-based equilibrium notions in general. Such coordination of entry decisions by firms might well violate existing antitrust legislation in practice.

In conclusion, the presence of network effects creates quite some novel features as far as the central problem of free entry is concerned, and some of these might call for some new antitrust legislation allowing for entry coordination (i.e. pre-play communication) between competitors, and create some new scope for useful coordinating activities by other actors, such as business associations. These conclusions reinforce the earlier findings that the start-up phase of a network industry might call for new forms of inter-firm cooperation.

8 Social welfare, consumer surplus and industry profits

This section studies the effects of an exogenous change in the number of firms on social welfare, consumer surplus and industry profits. As in the previous section, we continue to focus on the highest equilibrium outputs, $\overline{Z}_n$ and $\overline{x}_n$. Specifically, our aim is to give sufficient conditions that validate, for the highest equilibrium, the conventional wisdom that social welfare and consumer surplus increase with more competition, while industry profits decrease. Amir (2003) answers similar questions for standard Cournot competition, thus facilitating the corresponding comparisons.

We begin providing sufficient conditions for social welfare to increase with entry. Our initial assumptions, A1-A5, are consistent with $\overline{x}_n$ being increasing or decreasing in n, as reflected in Theorem 14. The next theorem shows the implications of these two possibilities on social welfare are quite different.

Theorem 18 *At the highest equilibrium, for any given n, $W_{n+1} \geq W_n$ if in addition to A1-A5 either one of the following conditions holds*

(i) $\int_0^{\overline{Z}_n} \left[P\left(t, \overline{Z}_{n+1}\right) - P\left(t, \overline{Z}_n\right)\right] dt \geq \overline{Z}_n \left[A\left(\overline{x}_{n+1}\right) - A\left(\overline{x}_n\right)\right]$*; or*

[23] Delgado and Moreno (2004) use coalition-proofness as a refinement to narrow down the set of supply function equilibria in an oligopolistic setting.

(ii) $\overline{x}_{n+1} \geq \overline{x}_n$.

Note that since $P_2(Z, S) > 0$, by A1, and $\overline{Z}_{n+1} \geq \overline{Z}_n$, by Theorem 13 (i), the left hand side of Condition (i) is always positive. So our theorem identifies the next two sufficient conditions: Welfare increases in the number of firms in the presence of diseconomies of scale ($A(.)$ is increasing) and decreasing per-firm output, or whenever per-firm output increases in n.

Network effects play a key role in these two conditions. As it is readily verified, they facilitate the first inequality by enlarging the left hand side of Condition (i). As seen earlier, network effects ease the conditions under which per-firm output increases in n, therefore facilitating Condition (ii).

The next result states that if marginal costs are constant, then social welfare, at the highest equilibrium, always increases with entry. Although this outcome follows as a direct implication of Theorem 18 (i), we include it as a separate result because it reflects the case most commonly analyzed in the existing literature.

Corollary 19 *In addition to A1-A5, assume the cost of production is linear, i.e.* $C(x) = cx$ *with* $c \geq 0$. *Then, at the highest equilibrium, social welfare always increases in the number of firms.*

We next study consumer surplus, for which our results differ markedly from their counterparts in the standard Cournot oligopoly.

Theorem 20 *At the highest equilibrium, for any given* n, $CS_{n+1} \geq CS_n$ *if, in addition to A1-A5, either (i)* $P\left(\overline{Z}_{n+1}, \overline{Z}_{n+1}\right) \leq P\left(\overline{Z}_n, \overline{Z}_n\right)$; *or (ii)* $P_{12}(Z, S) \leq 0$.

As a consequence of the so-called property of quasi-competitiveness, which under similar conditions holds in the standard Cournot game, Condition (i) is satisfied there. Example 4, at the end of the section, shows the opposite sometimes happens in network industries. Katz and Shapiro (1985) clearly explain why this surprising result might occur here: If the marginal consumer has a strong network externality, then the increment in the expected network size generated by the larger number of firms in the market, will increase his willingness to pay for the product above that of the average consumer. As a consequence, the firms will be able to raise the price by more than the increase in the average consumer's willingness to pay for the product and consumer's surplus will fall.

Our last theorem deals with industry profits. Like the previous two, it provides sufficient conditions for aggregate profits to increase after new entry.

Theorem 21 *At the highest equilibrium, for any given* n*,* $(n+1)\,\pi_{n+1} \geq n\pi_n$ *if, in addition to A1-A5, either (i)* $P\left(\overline{Z}_{n+1}, \overline{Z}_{n+1}\right) - P\left(\overline{Z}_n, \overline{Z}_n\right) \geq A\left(\overline{x}_{n+1}\right) - A\left(\overline{x}_n\right)$*; or (ii) the conditions of Theorem 16 (i) are satisfied.*

The proof of this result is quite simple. Relying on the fact that the highest selection of the equilibrium industry output increases in n, it simply says that, if the overall effect on the market price is larger than the change of the average cost of production, then industry profits increase. The second statement follows as a simple corollary of Theorem 16 (i).

Corollary 22 *In addition to A1-A5, assume the cost of production is linear, i.e.* $C(x) = cx$ *with* $c \geq 0$*. Then, at the highest equilibrium, industry profits increase in the number of firms if market price increases after new entry.*

The last result follows directly from Theorem 21 (i), as linear cost implies constant average cost.

Example 4 ends this section. It illustrates how, at the highest equilibrium industry output, consumer surplus and industry profits might decrease and increase, respectively, after new entry.

Example 4. Consider the symmetric Cournot oligopoly with no production costs and inverse demand function given by

$$P(Z,S) = \max\{a - Z/S^3, 0\} \text{ where } Z, S \in [0, nK] \text{ and } a, K > 1.$$

The reaction function of any given firm is

$$x(y,S) = \begin{cases} \max\left\{\left(aS^3 - y\right)/2, 0\right\} & \text{if } \left(aS^3 - y\right)/2 < K \\ K & \text{if } \left(aS^3 - y\right)/2 \geq K \end{cases}.$$

Thus, we have three possible equilibria

$$Z_n = \left\{0, \sqrt{(n+1)/(na)}, nK\right\}.$$

From a simple computation, consumer surplus is zero at the smallest equilibrium and, assuming $a \geq 1/(nK)^2$, it equals the following expression at the highest one

$$CS_n = 1/(2nK)$$

Since this expression is decreasing in n, consumer surplus decreases after new entry for the highest equilibrium. This result is possible because Conditions (i) and (ii) in Theorem 20 are not satisfied, i.e. the market price at the highest equilibrium increases in n and $P_{12}(Z,S) = 3/S^4 > 0$ for all Z, S.

Note that the opposite is true for aggregate profits. The following expression shows that they increase in n at the highest equilibrium

$$n\pi_n = nK\left[a - 1/(nK)^2\right].$$

As Corollary 19 states, we next show that social welfare, at the highest equilibrium, increases in n

$$W_n = anK - 1/(2nK).$$

These results point out some of the relevant differences between Cournot competition with and without network effects in terms of industry performance. □

9 Proofs

This section provides the proofs for all the results of the paper, and also contains the statements and proofs of some useful intermediate results not given in the body of the paper.

The proof of Lemma 1 calls for an intermediate result.

Lemma 23 *Assume A1-A5 hold. Then $\widetilde{\pi}(Z, y, S)$ has the strict single-crossing property in $(Z; S)$.*

Proof of Lemma 23

To prove this result, first note that $\Delta_2(Z, S) > 0$ if and only if $\partial^2 \log P(Z, S)/\partial Z \partial S > 0$. We show that this condition implies that $\widetilde{\pi}(Z, y, S)$ has the strict single-crossing property in $(Z; S)$, i.e. that for any $Z > Z'$ and $S > S'$,

$$\widetilde{\pi}\left(Z, y, S'\right) \geq \widetilde{\pi}\left(Z', y, S'\right) \Longrightarrow \widetilde{\pi}(Z, y, S) > \widetilde{\pi}\left(Z', y, S\right). \tag{12}$$

Since $\partial^2 \log P(Z, S)/\partial Z \partial S > 0$ we have $\log P(Z, S) - \log P(Z', S) > \log P(Z, S') - \log P(Z', S')$, or

$$\frac{P(Z, S)}{P(Z', S)} > \frac{P(Z, S')}{P(Z', S')}. \tag{13}$$

The left hand side of (12) can be rewritten as

$$(Z - y) P\left(Z, S'\right) - C(Z - y) \geq \left(Z' - y\right) P\left(Z', S'\right) - C\left(Z' - y\right). \tag{14}$$

Combining (13) and (14), we get

$$(Z - y) P(Z, S) \frac{P(Z', S')}{P(Z', S)} - C(Z - y) > \left(Z' - y\right) P\left(Z', S'\right) - C\left(Z' - y\right). \tag{15}$$

Multiplying both sides of (15) by $P(Z', S)/P(Z', S')$ we obtain

$$(Z-y)\, P(Z,S) - \frac{P(Z',S)}{P(Z',S')} C(Z-y) > \left(Z'-y\right) P\left(Z',S\right) - \frac{P(Z',S)}{P(Z',S')} C\left(Z'-y\right). \tag{16}$$

By A1, $P(Z', S)/P(Z', S') > 1$ and, by A2, $C(Z-y) \geq C(Z'-y)$. Thus, (16) implies

$$(Z-y)\, P(Z,S) - C(Z-y) > \left(Z'-y\right) P\left(Z',S\right) - C\left(Z'-y\right), \tag{17}$$

which is just the right hand side of (12). Hence, (12) holds. □

Proof of Lemma 1

Since $\partial^2 \widetilde{\pi}(Z, y, S)/\partial Z \partial y = \Delta_1(Z, y) > 0$, by A4, the maximand in (2) has strictly increasing differences in (Z, y). Furthermore, the feasible correspondence $(y, S) \longrightarrow [y, y+K]$ is ascending in y. Then, by Topkis's theorem [Theorem A.1, Appendix], every selection from the argmax of $\widetilde{\pi}(Z, y, S)$, $\widetilde{Z}(y, S)$, increases in y.

By Lemma 23, $\widetilde{\pi}(Z, y, S)$ has the strict single-crossing property in $(Z; S)$. In addition, the feasible correspondence $(y, S) \longrightarrow [y, y+K]$ does not depend on S. Then, by [Theorem A.2, Appendix] due to Milgrom and Shannon (1994), every selection from the argmax of $\widetilde{\pi}(Z, y, S)$, $\widetilde{Z}(y, S)$, is also increasing in S. □

Proof of Theorem 2

The following mapping, which can be thought of as a normalized cumulative best-response, is the key element in dealing with symmetric equilibria for any n[24]

$$\begin{aligned} B_n : [0, (n-1)K] \times [0, nK] &\longrightarrow 2^{[0,(n-1)K]\times[0,nK]} \\ (y, S) &\longrightarrow \left[\frac{n-1}{n}\left(x'+y\right), x'+y\right] \end{aligned}$$

where x' denotes a best-response output level by a firm to a joint output y by the other $(n-1)$ firms, given S. It is readily verified that the (set-valued) range of B_n is as given, i.e. if $x' \in [0, K]$ and $y \in [0, (n-1)K]$, then $((n-1)/n)(x'+y) \in [0, (n-1)K]$ and $x'+y \in [0, nK]$. Also, a fixed point of B_n is easily seen as a symmetric equilibrium, for it must satisfy both $\widehat{y} = ((n-1)/n)(\widehat{x}'+\widehat{y})$, or $\widehat{x}' = \widehat{y}/(n-1)$, and $\widehat{S} = \widehat{x}' + \widehat{y}$, which says that the responding firm produces as much as each of the other $(n-1)$ firms and the expected size of the network is fulfilled at equilibrium.

By Lemma 1 we know that every selection of $\widetilde{Z}(y, S)$ increases in y and S. Hence, for any fixed $n \in N$, every selection of B_n increases in (y, S), so that by Tarski's fixed point theorem

[24] See Amir and Lambson (2000) and Kwon (2007).

[Theorem A.3, Appendix], it has a fixed point. As argued before, a fixed point of B_n is a symmetric equilibrium. This proves the first statement of Theorem 2. We next show that no asymmetric equilibria exists.

To this end, it suffices to show that the correspondence $\widetilde{Z}(y,S)$ is strictly increasing (in the sense that all its selections are strictly increasing) in y, for each S. Thus, for all possible S, to each $Z' \in \widetilde{Z}(y,S)$ corresponds (at most) one y such that $Z' = x' + y$ with Z' being a best-response to y and S. In other words, for each equilibrium output Z', each firm must be producing the same $x' = Z' - y$, with $y = (n-1)x'$.

A4 implies that $\partial\widetilde{\pi}(Z,y,S)/\partial Z$ is strictly increasing in y, a property slightly stronger than strictly increasing differences in (Z,y). By Topkis (1998), Theorem 2.8.5 on p. 79, this property implies that $\widetilde{Z}(y,S)$ is strictly increasing in y for each S, whenever $\widetilde{Z}(y,S)$ is interior.[25] The second statement in Theorem 2 follows because, as argued in the previous paragraph, this condition guarantees no asymmetric equilibria exist. $\square$

The next proof, from Amir and Lambson (2000), is included for completeness only.

Proof of Lemma 3

To show Part (i) consider the mapping

$$\begin{aligned} T_n : [0,(n-1)K] &\longrightarrow 2^{[0,(n-1)K]} \\ y &\longrightarrow \left[\frac{n-1}{n}\left(x'+y\right)\right]. \end{aligned} \tag{18}$$

The proof of existence and the fact that no asymmetric equilibrium exists follow as a simply corollary of the proof of Theorem 2, thus we omit it.

We next show that if A5 is also satisfied, the extremal selections of $z_n(S)$, $\overline{z}_n(S)$ and $\underline{z}_n(S)$, increase in S. We know, by Topkis's theorem, that the maximal and minimal selections of T_n denoted, respectively, $\overline{T}_n$ and $\underline{T}_n$, exist. Furthermore, the largest value of $z_n(S)$, $\overline{z}_n(S)$, constitutes the largest fixed point of $\overline{T}_n$. Under A5 we know, by Lemma 1, that every selection of $\widetilde{Z}(y,S)$ increases in S. Then the largest fixed point of $\overline{T}_n$, $\overline{z}_n(S)$, is also increasing in S [Theorem A.4, Appendix]. A similar argument, using the selection $\underline{T}_n$, establishes that $\underline{z}_n(S)$ is increasing in S. This ends the proof of Part (ii).

To prove Part (iii), we show that adding A6 leads to $z_n(.)$ being a single-valued and continuous function. From Amir (1996a), Theorem 2.1, we know that the best-response correspondence $\widetilde{x}(y,S)$,

[25] This result was proved in Amir (1996) and Edlin and Shannon (1998).

as defined in (1), is nonincreasing given that $P(Z,S)$ is log-concave in Z. In addition, since every selection of $\widetilde{Z}(y,S)$ increases in y (Lemma 1) and $\widetilde{Z}(y,S) = \widetilde{x}(y,S) + y$, it follows that every selection of $\widetilde{x}(y,S)$ has all its slopes bounded below by -1. Altogether then, all the slopes of every selection of $\widetilde{x}(y,S)$ lie in $[-1,0]$. This leads to the uniqueness of Cournot equilibrium through a well-known argument, a proof of which is given in Amir and Lambson (2000), Theorem 2.3. Hence, $z_n(.)$ is single-valued. Since $z_n(.)$ is also u.h.c. as a correspondence, due to firms' payoffs being continuous in S, the conclusion follows. □

Proof of Lemma 4

By definition, an industry output of 0 is a FECE if $0 \in \widetilde{x}(0,0)$. This holds if and only if we have

$$\begin{aligned} \pi(0,0,0) &\geq \pi(x,0,0) \\ 0 &\geq xP(x,0) - C(x) \\ C(x) &\geq xP(x,0) \end{aligned}$$

for all $x \in [0,K]$. This proves our first statement. The second one follows because all the steps are independent of the number of firms in the market. □

The proof of Theorem 5 calls for several intermediate results, which will turn out to be useful for some other proofs as well. We first state sufficient conditions under which an increasing selection of $z_n(S)$ is differentiable for almost all S, and give a specific functional form for its slope. We then show that when 0 is part of the equilibrium set, then $z_n(0)$ is single-valued and right-continuous.

Lemma 24 *Assume A1-A5 are satisfied. Let $\widehat{z}_n$ be an increasing selection of $z_n(S)$, such that $\widehat{z}_n(S) \in (0, nK)$. Then $\widehat{z}_n(S)$ is differentiable for almost all S, and its slope is given by*

$$\frac{\partial \widehat{z}_n(S)}{\partial S} = \frac{-n\left\{P_1(\widehat{z}_n,S)P_2(\widehat{z}_n,S) - \left[P(\widehat{z}_n,S) - C'(\widehat{z}_n/n)\right]P_{12}(\widehat{z}_n,S)\right\}}{(n+1)\left[P_1(\widehat{z}_n,S)\right]^2 - n\left[P(\widehat{z}_n,S) - C'(\widehat{z}_n/n)\right]P_{11}(\widehat{z}_n,S) - P_1(\widehat{z}_n,S)C''(\widehat{z}_n/n)} \tag{19}$$

where $\widehat{z}_n$ stands for $\widehat{z}_n(S)$.

Proof of Lemma 24

If $\widehat{z}_n(S)$ is interior, it must satisfy the first order condition

$$P(\widehat{z}_n,S) + (\widehat{z}_n/n)\,P_1(\widehat{z}_n,S) - C'(\widehat{z}_n/n) = 0 \tag{20}$$

where $\widehat{z}_n$ stands for $\widehat{z}_n(S)$. Multiplying both sides of (20) by n

$$nP(\widehat{z}_n, S) + z_n P_1(\widehat{z}_n, S) - nC'(\widehat{z}_n/n) = 0. \tag{21}$$

Since $\widehat{z}_n(S)$ is increasing, it is differentiable almost everywhere (w.r.t. Lebesgue measure) and

$$\frac{\partial \widehat{z}_n(S)}{\partial S} = \frac{-\left[nP_2(\widehat{z}_n, S) + \widetilde{z}_n P_{12}(\widehat{z}_n, S)\right]}{(n+1)\, P_1(\widehat{z}_n, S) + \widehat{z}_n P_{11}(\widehat{z}_n, S) - C''(\widehat{z}_n/n)}. \tag{22}$$

Substituting $\widehat{z}_n(S)$ by its implicit value in (20), and multiplying the numerator and the denominator by $P_1(z_n, S)$, we obtain (19). □

Proof of Lemma 6

We first show that if $0 \in z_n(0)$, then $0 = z_n(0)$, i.e., $z_n(0)$ is a singleton. By Lemma 4 we know that $0 \in z_n(0)$ if and only if

$$xP(x, 0) \leq C(x) \text{ for all } x \in [0, K]. \tag{23}$$

Since $P_1(Z, S) < 0$ by A1, it follows from (23) that $xP(x + y, 0) < C(x)$ for all $x \in (0, K]$ and all $y > 0$. Hence, 0 is a dominant strategy in the standard Cournot game given $S = 0$. This proves that $z_n(0)$ is single-valued.

The fact that $P(0, 0) = C'(0)$ ensures the trivial outcome is an interior equilibrium. To show (5), take any sequence $S_k \downarrow 0$ such that $\widehat{z}_n$ is differentiable at S_k for all k (this is possible since the set of points of differentiability of an increasing function forms a dense subset of its domain). Since $\widehat{z}_n$ is increasing, it has left and right limits at every point, so the limit $\lim_{k\to\infty} \widehat{z}_n(S_k)$ exists. Since $z_n(.)$ is u.h.c. (see the proof of Lemma 3), $\lim_{k\to\infty} \widehat{z}_n(S_k) \in z_n(0) = \{0\}$, so that by the earlier part of this proof, $\lim_{k\to\infty} \widehat{z}_n(S_k) = 0$.

Now consider (22) with $S = S_k$. By Assumption A1 and the fact that $\lim_{k\to\infty} \widehat{z}_n(S_k) = 0$, the right-hand side of (22) is right-continuous in S at 0. Taking limits as $k \longrightarrow \infty$, (5) follows. Since this argument is clearly independent of the particular (increasing) selection $\widehat{z}_n$ and of the sequence (S_k) chosen, $\partial z_n(S)/\partial S|_{S=0}$ is single-valued, continuous at 0, and given by (5).

The fact that $z_n'(0) = 0$ if the trivial equilibrium in not interior follows directly from our previous arguments, thus we omit this proof. □

We next show that, for all $S \in [0, nK]$, any argmax of a fictitious objective function $\Pi(Z, S)$ is an element of $z_n(S)$. Note that, given S, $\Pi(Z, S)$ may be viewed as a weighted combination of industry profits and welfare, with respective weights $\frac{1}{n}$ and $\frac{n-1}{n}$, as constructed by Bergstrom and Varian (1985) for standard Cournot.

Lemma 25 *Assume A1-A5 are satisfied and* $C(.)$ *is convex. Define*

$$\Pi(Z,S) \triangleq \frac{n-1}{n}\int_0^Z P(t,S)\,dt + \frac{Z}{n}P(Z,S) - nC(Z/n).$$

Given any $n \in N$ *and* $S \in [0, nK]$, *if* $Z' \in \arg\max\{\Pi(Z,S) : 0 \leq Z \leq nK\}$ *then* $Z' \in z_n(S)$.

Proof of Lemma 25

Assume Z^* is an argmax of $\Pi(Z,S)$, we need to show Z^* corresponds to the industry output of a symmetric Cournot equilibrium with exogenous S. Let $Z^* = x^* + y^*$, with $x^* = Z^*/n$ and $y^* = (n-1)x^*$, and consider $Z' = x' + y^*$, with $x' \in [0,K]$. Then x' denotes a possible deviation of a given firm from its equilibrium output x^*. We next show this unilateral deviation is never profitable.

Since Z^* is a maximizer of $\Pi(Z,S)$, then $\Pi(Z^*,S) \geq \Pi(Z',S)$, which is equivalent to say

$$\begin{aligned}
&\frac{n-1}{n}\int_0^{x^*+y^*} P(t,S)\,dt + x^*P(x^*+y^*,S) - nC(x^*) \geq \\
&\frac{(n-1)}{n}\int_0^{x'+y^*} P(t,S)\,dt + \frac{(x'+y^*)}{n}P\left(x'+y^*,S\right) - nC\left(\frac{x'+y^*}{n}\right)
\end{aligned} \tag{24}$$

Then we have

$$\begin{aligned}
& x^*P(x^*+y^*,S) - C(x^*) \\
\geq\ & \frac{n-1}{n}\int_0^{x'+y^*} P(t,S)\,dt + \frac{(x'+y^*)}{n}P\left(x'+y^*,S\right) - nC\left(\frac{x'+y^*}{n}\right) \\
& -\frac{n-1}{n}\int_0^{x^*+y^*} P(t,S)\,dt + (n-1)C(x^*) \\
\geq\ & \frac{n-1}{n}\int_{x^*+y^*}^{x'+y^*} P(t,S)\,dt + \frac{(x'+y^*)}{n}P\left(x'+y^*,S\right) - C\left(x'\right) \\
\geq\ & \frac{(n-1)(x'-x^*)}{n}P\left(x'+y^*,S\right) + \frac{(x'+y^*)}{n}P\left(x'+y^*,S\right) - C\left(x'\right) \\
=\ & x'P\left(x'+y^*,S\right) - C\left(x'\right).
\end{aligned}$$

The first inequality follows from (24), after rearranging terms. The second one holds as we assumed $C(.)$ is convex (and $y^* = (n-1)x^*$), and the last one by A1, $P_1(Z,S) < 0$. Since x' is arbitrary, this argument shows that x^* is a symmetric Cournot equilibrium. □

Proof of Theorem 5

Part (i) holds because, if the trivial outcome (zero output) is not part of the equilibrium set, Theorem 2 guarantees there is a FECE with strictly positive industry output.

Parts (ii) and (iii) are both based on the following argument. By Lemma 3, the maximal and minimal selections of $z_n(S)$, $\overline{z}_n(S)$ and $\underline{z}_n(S)$, increase in S. Assume, for the moment, there exists an $S' \in (0, nK]$ such that $\overline{z}_n(S') \geq S'$. If we restrict attention to the values of S in $[S', nK]$, it follows that $\overline{z}_n(S) \in [S', nK]$ because $\overline{z}_n(.)$ is increasing and $\overline{z}_n(S') \geq S'$. Therefore, for all $S \in [S', nK]$, $\overline{z}_n(S)$ is an increasing function that maps $[S', nK]$ into itself. Hence, by Tarski's fixed point theorem [Theorem A.3, Appendix], there is an $S' \leq S'' \leq nK$ such that $\overline{z}_n(S'') = S''$. Since this condition implies $\overline{z}_n(S'')$ is a strictly positive FECE, the existence of a nontrivial equilibrium reduces to showing there is at least one $S \in (0, nK]$ for which an element of $z_n(S)$ is above S, i.e. $\overline{z}_n(S) \geq S$.

To prove Part (ii), we show $z_n'(0) > 1$. Using Lemma 6, $z_n'(0) > 1$ if, given $P_1(0,0) + P_2(0,0) > 0$,

$$n > \left[-P_1(0,0) + C''(0)\right] / \left[P_1(0,0) + P_2(0,0)\right].$$

Then the existence of a nontrivial FECE follows by the argument in the previous paragraph, as Lemma 6 and the property $z_n'(0) > 1$, imply there exists a small $\varepsilon > 0$ for which $\overline{z}_n(\varepsilon) > \varepsilon$. This completes the proof of Part (ii).

Condition (4) in Part (iii) is equivalent to say there is some $S \in (0, nK]$ and some $\widehat{Z} \geq S$ for which $n\left[\Pi\left(\widehat{Z}, S\right) - \Pi(Z, S)\right] \geq 0$ for all $Z \leq S$. As a consequence, the largest argmax of $\Pi(Z, S)$ must be larger than S. Call this argmax Z'. Our proof follows because $Z' \in z_n(S)$, by Lemma 25, and this ensures there is an $S \in (0, nK]$ for which an element of $z_n(S)$ is higher than S. $\square$

Proof of Theorem 7

Under A1-A6 we know, by Lemma 3, that $z_n(.)$ is a single-valued, continuous and increasing function. The fact that $f(Z, n)$ is equal to its slope along the diagonal, follows from a stronger version of Lemma 24 as follows. Consider (20) with $z_n(S)$ instead of $\widehat{z}_n(S)$. By the implicit function theorem, $\partial z_n(S)/\partial S$ exists at every S and is given by (19) with $\widehat{z}_n(S)$ replaced by $z_n(S)$. Evaluating this along the diagonal $Z = S$, we see that it is equal to $f(Z, n)$.

The uniqueness result now follows directly from the assumption $f(Z, n) < 1$ for all Z, since any two adjacent fixed points of $z_n(S)$ must include one for which $f(Z, n) \geq 1$. $\square$

Proof of Proposition 10

By Lemma 3, $\underline{z}_n(.)$ and $\overline{z}_n(.)$ are increasing.

To prove Part (i), first assume 0 is not an equilibrium. Then $\underline{z}_n(0) > 0$ and the orbit in (8) with $\widehat{z}_n = \underline{z}_n$ and $S_0 = 0$ (or S_0 near 0) must converge to the smallest fixed point of $\underline{z}_n(.)$, which is a strictly positive equilibrium by (the successive approximation part of) Tarski's fixed point theorem. Hence orbits with higher values of S_0 will converge to non-zero fixed points of $\underline{z}_n(.)$.

Next, assume Condition (ii) of Theorem 5 holds. Then we know that $\underline{z}_n(S) > S$ for S small enough (cf. Lemma 6). So any orbit with S_0 near 0 converges to the smallest fixed point of $\underline{z}_n(.)$ with strictly positive output, and orbits with higher values of S_0 are as in the previous step.

To prove Part (ii), Condition (iii) of Theorem 5 ensures there is an $S_0 \in (0, nK]$ for which $\overline{z}_n(S_0) \geq S_0$. By Tarski's Theorem applied to $\overline{z}_n$ mapping $[S_0,nK]$ to itself, the orbit starting at S_0 must converge to a strictly positive equilibrium.

To prove Part (iii), note that since 0 is a FECE, there can be no other FECE by Theorem 7. Hence, every orbit from any S_0 is a decreasing sequence to 0. □

Proof of Theorem 12

To prove Part (i), we use the mapping (18) in the proof of Lemma 3. We know, by Topkis's theorem, that the maximal and minimal selections of T_n, $\overline{T}_n$ and $\underline{T}_n$, exist. Furthermore, the largest value of $z_n(S)$, $\overline{z}_n(S)$, constitutes the largest fixed point of $\overline{T}_n$. Since $(n-1)/n$ increases in n every selection of T_n is increasing. Then the largest fixed point of $\overline{T}_n$, $\overline{z}_n(S)$, is also increasing in n [Theorem A.4, Appendix]. A similar argument, using the selection $\underline{T}_n$, establishes that $\underline{z}_n(S)$ is increasing in n.

The second statement of Part (i) follows directly from Definition 11 and the fact that $\underline{z}_n(S)$ and $\overline{z}_n(S)$ shift up as n increases.

To prove Part (ii), observe that if the trivial equilibrium holds and $P_1(0,0) + P_2(0,0) \leq 0$, then, by Lemma 6, $\overline{z}'_n(0) < 1 \ \forall \ n$, so that 0 is a stable equilibrium $\forall \ n$. This ends our proof. □

Proof of Theorem 13

The maximal and minimal selections of B_n (as defined in the proof of Theorem 2) denoted, respectively, $\overline{B}_n$ and $\underline{B}_n$, exist by Topkis's theorem. Furthermore, the largest equilibrium values of y_n and Z_n , $(\overline{y}_n, \overline{Z}_n)$, constitute the largest fixed point of $\overline{B}_n$. Since $(n-1)/n$ is increasing in n, $\overline{B}_n$ is increasing in n for all (y, S). Since $\overline{B}_n$ is also increasing in both y and S, the largest fixed point of $\overline{B}_n$, $(\overline{y}_n, \overline{Z}_n)$, is also increasing in n (see Milgrom and Roberts, 1990). A similar argument, using the selection $\underline{B}_n$, establishes that $\left(\underline{y}_n, \underline{Z}_n\right)$ increases in n as well. This shows part (i).

Part (ii) follows directly from Part (i) since $dP(Z,Z)/dz = P_1(Z,Z) + P_2(Z,Z)$. □

Proof of Theorem 14

Consider the following mapping

$$\begin{aligned} M_n : [0, nK] &\longrightarrow 2^{[0,K]} \\ Z &\longrightarrow \widetilde{x} = \left\{ x : P(Z,Z) + xP_1(Z,Z) - C'(x) = 0 \right\}. \end{aligned} \tag{25}$$

Then M_n maps industry output into the solution of a fictitious first order condition, which coincides with that of an interior FECE when $x = Z/n$ and $Z = Z_n$.

Totally differentiating this first order condition with respect to n, we have

$$\{P_1(Z,Z) + P_2(Z,Z) + \widetilde{x}\,[P_{11}(Z,Z) + P_{12}(Z,Z)]\} \frac{dz}{dn} = 0. \tag{26}$$

Substituting in (26) $\widetilde{x}$ by $[C'(Z/n) - P(Z,Z)]/P_1(Z,Z)$, and rearranging terms, we get

$$-\frac{1}{P_1(Z,Z)} \left\{ \left[P(Z,Z) - C'(Z/n) \right] [P_{11}(Z,Z) + P_{12}(Z,Z)] - P_1(Z,Z)\,[P_1(Z,Z) + P_2(Z,Z)] \right\} \frac{dz}{dn} = 0. \tag{27}$$

Substituting $g(Z)$ from (9) into (27), we get

$$-\frac{1}{P_1(Z,Z)} g(Z) \frac{dz}{dn} = 0. \tag{28}$$

By A1, $P_1(Z,Z) < 0$. Also, by Theorem 13 (i), the extremal equilibrium industry outputs increase in n. Then, if $g(Z) \geq (\leq) 0$ over $\left[\overline{Z}_n, \overline{Z}_{n+1}\right]$, the mapping M_n increases (decreases) in n at the largest equilibrium industry output. Theorem 14 follows because if M_n increases (decreases) in n at the largest equilibrium industry output, then $\overline{x}_n$ also increases (decreases) with this parameter. By a similar argument it can be shown that this is also true for $\underline{x}_n$. □

Proof of Corollary 15

Inequality 10 equals function $g(Z)$ when the firms face no cost of production. Then the first claim follows directly from Theorem 14 (i).

The first term in the left hand side of (10) is always positive by A5. As the log-convexity of $P(Z,S)$ in Z guarantees the second term is also positive, this is a sufficient condition for the required inequality. □

Proof of Theorem 16

Consider the following inequalities

$$\begin{aligned}\pi_{n+1} &= \overline{x}_{n+1}P\left(\overline{x}_{n+1}+\overline{y}_{n+1},\overline{Z}_{n+1}\right)-C\left(\overline{x}_{n+1}\right)\\ &\geq \overline{x}_{n}P\left(\overline{x}_{n}+\overline{y}_{n+1},\overline{Z}_{n+1}\right)-C\left(\overline{x}_{n}\right)\\ &\geq \overline{x}_{n}P\left(\overline{x}_{n+1}+\overline{y}_{n+1},\overline{Z}_{n+1}\right)-C\left(\overline{x}_{n}\right)\\ &\geq \overline{x}_{n}P\left(\overline{x}_{n}+\overline{y}_{n},\overline{Z}_{n}\right)-C\left(\overline{x}_{n}\right)\\ &= \pi_{n}.\end{aligned}$$

The first inequality follows by the Cournot equilibrium property. The second one is from $\overline{x}_{n+1} \geq \overline{x}_n$ and A1. (The fact that $\overline{x}_{n+1} \geq \overline{x}_n$ here follows by Theorem 14 (i) because we assumed all its required conditions are satisfied.) The third inequality follows because our assumptions imply $P\left(\overline{Z}_{n+1},\overline{Z}_{n+1}\right) \geq P\left(\overline{Z}_{n},\overline{Z}_{n}\right)$. Therefore, $\overline{\pi}_{n+1} \geq \overline{\pi}_n$. By a similar argument it can be shown that this is also true for the equilibrium per-firm profits evaluated at the minimal equilibrium outputs. This shows Part (i).

We omit the proof of Part (ii) as it is almost identical to the previous one. □

Proof of Corollary 17

If $P_{11}(Z,Z)+P_{12}(Z,Z)=0$, then $g(Z)=-P_1(Z,Z)\left[P_1(Z,Z)+P_2(Z,Z)\right]$. By A1, $P_1(Z,Z)<0$. Then the sign of $g(Z)$ is equal to the sign of $P_1(Z,Z)+P_2(Z,Z)$, and Corollary 17 (i) and (ii) follow by Theorems 14 (i) and 16 (i), respectively. □

Proof of Theorem 18

To show Part (i) consider

$$\begin{aligned}W_{n+1}-W_n &= \int_0^{\overline{Z}_{n+1}} P\left(t,\overline{Z}_{n+1}\right)dt-\overline{Z}_{n+1}A\left(\overline{x}_{n+1}\right)-\left[\int_0^{\overline{Z}_n} P\left(t,\overline{Z}_n\right)dt-\overline{Z}_nA\left(\overline{x}_n\right)\right]\\ &\geq \int_0^{\overline{Z}_n} P\left(t,\overline{Z}_{n+1}\right)dt-\overline{Z}_nA\left(\overline{x}_{n+1}\right)-\left[\int_0^{\overline{Z}_n} P\left(t,\overline{Z}_n\right)dt-\overline{Z}_nA\left(\overline{x}_n\right)\right]\\ &\geq 0.\end{aligned}$$

The first inequality follows because $P\left(t,\overline{Z}_{n+1}\right)-A\left(\overline{x}_{n+1}\right)\geq 0$ for all $t\leq \overline{Z}_{n+1}$, and $\overline{Z}_{n+1}\geq \overline{Z}_n$ by Theorem 13 (i). The second inequality holds by the assumed conditions.

To show Part (ii) let us define $V_n(x,S)=\int_0^{nx}P(t,S)\,dt-nC(x)$. Notice $V_n(x,S)$ is concave in

x since $n\left[nP_1\left(nx,S\right)-C''\left(x\right)\right]<0$ by both A1 and A4. In addition,

$$\begin{aligned}\int_0^{\overline{Z}_{n+1}} P\left(t,\overline{Z}_{n+1}\right)dt &= \int_0^{n\overline{x}_{n+1}} P\left(t,\overline{Z}_{n+1}\right)dt+\int_{n\overline{x}_{n+1}}^{\overline{Z}_{n+1}} P\left(t,\overline{Z}_{n+1}\right)dt \\ &\geq \int_0^{n\overline{x}_{n+1}} P\left(t,\overline{Z}_{n+1}\right)dt+\overline{x}_{n+1}P\left(\overline{Z}_{n+1},\overline{Z}_{n+1}\right) \qquad (29)\end{aligned}$$

where the inequality follows by A1. The following steps show our result

$$\begin{aligned}W_{n+1}-W_n &= \int_0^{(n+1)\overline{x}_{n+1}} P\left(t,\overline{Z}_{n+1}\right)dt-(n+1)\,C\left(\overline{x}_{n+1}\right)-\left[\int_0^{n\overline{x}_n} P\left(t,\overline{Z}_n\right)dt-nC\left(\overline{x}_n\right)\right] \\ &\geq \pi_{n+1}+\int_0^{n\overline{x}_{n+1}} P\left(t,\overline{Z}_{n+1}\right)dt-nC\left(\overline{x}_{n+1}\right)-\left[\int_0^{n\overline{x}_n} P\left(t,\overline{Z}_n\right)dt-nC\left(\overline{x}_n\right)\right] \\ &\geq \pi_{n+1}+\int_0^{n\overline{x}_{n+1}} P\left(t,\overline{Z}_{n+1}\right)dt-nC\left(\overline{x}_{n+1}\right)-\left[\int_0^{n\overline{x}_n} P\left(t,\overline{Z}_{n+1}\right)dt-nC\left(\overline{x}_n\right)\right] \\ &= \pi_{n+1}+V_n\left(\overline{x}_{n+1},\overline{Z}_{n+1}\right)-V_n\left(\overline{x}_n,\overline{Z}_{n+1}\right) \\ &\geq \pi_{n+1}+\left[\partial V_n\left(\overline{x}_{n+1},\overline{Z}_{n+1}\right)/\partial x\right]\left(\overline{x}_{n+1}-\overline{x}_n\right) \\ &= \pi_{n+1}+n\left[P\left(n\overline{x}_{n+1},\overline{Z}_{n+1}\right)-C'\left(\overline{x}_{n+1}\right)\right]\left(\overline{x}_{n+1}-\overline{x}_n\right) \\ &\geq \pi_{n+1}+n\left[P\left((n+1)\,\overline{x}_{n+1},\overline{Z}_{n+1}\right)-C'\left(\overline{x}_{n+1}\right)\right]\left(\overline{x}_{n+1}-\overline{x}_n\right) \\ &\geq 0.\end{aligned}$$

The first inequality follows from inequality (29), the second one by A1 and Theorem 13 (i) and the third one by the concavity of $V_n\left(x,S\right)$ in x. The fourth inequality holds by A1 and because we assumed $\overline{x}_{n+1}\geq\overline{x}_n$, and the last one by the Cournot property. This completes our proof. □

Proof of Corollary 19

If the cost of production is linear, the right hand side of the required condition in Theorem 18 (i) is zero. Its left hand side is always positive because $\overline{Z}_{n+1}\geq\overline{Z}_n$ and, by A1, $P_2\left(Z,S\right)>0$. Our result follows because these two facts ensure Theorem 18 (i) is satisfied. □

Proof of Theorem 20

The proof of Part (i) follows directly from Theorem 13 (i).

The following steps prove Part (ii)

$$\begin{aligned}
CS_{n+1}-CS_n &= \int_0^{\overline{Z}_{n+1}}\left[P\left(t,\overline{Z}_{n+1}\right)-P\left(\overline{Z}_{n+1},\overline{Z}_{n+1}\right)\right]dt-\int_0^{\overline{Z}_n}\left[P\left(t,\overline{Z}_n\right)-P\left(\overline{Z}_n,\overline{Z}_n\right)\right]dt \\
&\geq \int_0^{\overline{Z}_n}\left[P\left(t,\overline{Z}_{n+1}\right)-P\left(\overline{Z}_{n+1},\overline{Z}_{n+1}\right)\right]dt-\int_0^{\overline{Z}_n}\left[P\left(t,\overline{Z}_n\right)-P\left(\overline{Z}_n,\overline{Z}_n\right)\right]dt \\
&= \overline{Z}_n\left[P\left(\overline{Z}_n,\overline{Z}_n\right)-P\left(\overline{Z}_{n+1},Z_n\right)\right] \\
&\quad -\int_0^{\overline{Z}_n}\left\{\left[P\left(\overline{Z}_{n+1},\overline{Z}_{n+1}\right)-P\left(\overline{Z}_{n+1},\overline{Z}_n\right)\right]-\left[P\left(t,\overline{Z}_{n+1}\right)-P\left(t,\overline{Z}_n\right)\right]\right\}dt \\
&\geq \overline{Z}_n\left[P\left(\overline{Z}_n,\overline{Z}_n\right)-P\left(\overline{Z}_{n+1},Z_n\right)\right] \\
&\geq 0.
\end{aligned}$$

The first inequality follows directly from $P_1(Z,S)<0$ and Theorem 13 (i). The next step is obtained from the previous one by adding and subtracting $\int_0^{\overline{Z}_n}P\left(Z_{n+1},\overline{Z}_n\right)dt$, and rearranging terms. To justify the second inequality notice that $P_{12}(Z,S)\leq 0$ is sufficient for

$$\int_0^{\overline{Z}_n}\left[P\left(t,\overline{Z}_{n+1}\right)-P\left(t,\overline{Z}_n\right)\right]dt\geq\int_0^{\overline{Z}_n}\left[P\left(\overline{Z}_{n+1},\overline{Z}_{n+1}\right)-P\left(\overline{Z}_{n+1},\overline{Z}_n\right)\right]dt.$$

Our last step is true since $P_1(Z,S)<0$.

Hence, $P_{12}(Z,S)\leq 0\ \forall\ Z,S\in[0,nK]$ is sufficient for $CS_{n+1}-CS_n\geq 0$, or $CS_{n+1}\geq CS_n$. □

Proof of Theorem 21

For an extremal equilibrium industry output, consider

$$\begin{aligned}
(n+1)\,\pi_{n+1}-n\pi_n &= \overline{Z}_{n+1}\left[P\left(\overline{Z}_{n+1},\overline{Z}_{n+1}\right)-A\left(\overline{x}_{n+1}\right)\right]-\overline{Z}_n\left[P\left(\overline{Z}_n,\overline{Z}_n\right)-A\left(\overline{x}_n\right)\right] \\
&\geq \overline{Z}_n\left[P\left(\overline{Z}_{n+1},\overline{Z}_{n+1}\right)-A\left(\overline{x}_{n+1}\right)\right]-\overline{Z}_n\left[P\left(\overline{Z}_n,\overline{Z}_n\right)-A\left(\overline{x}_n\right)\right] \\
&= \overline{Z}_n\left\{\left[P\left(\overline{Z}_{n+1},\overline{Z}_{n+1}\right)-P\left(\overline{Z}_n,\overline{Z}_n\right)\right]-\left[A\left(\overline{x}_{n+1}\right)-A\left(\overline{x}_n\right)\right]\right\}.
\end{aligned}$$

Since $P\left(\overline{Z}_{n+1},\overline{Z}_{n+1}\right)-A\left(\overline{x}_{n+1}\right)\geq 0$, the inequality follows by Theorem 13 (i). The first part of Theorem 21 simply says that if the last function is positive, then $(n+1)\,\pi_{n+1}\geq n\pi_n$. This shows Part (i). Part (ii) follows directly from Theorem (16) (i) so we omit it. □

Proof of Corollary 22

This result follows directly from Theorem 21 (i), because linear cost implies constant average cost of production. □

APPENDIX

In an attempt to make this paper self-contained, we provide a summary of all lattice-theoretic notions and results used here. Since this paper deals with real decision and parameter spaces, every theorem that follows is a special case of the original one (see Topkis, 1998).

A function F: $R_+^2 \to R$ is supermodular if, for $x_1 \geq x_2$, $y_1 \geq y_2$,

$$F(x_1, y_1) - F(x_2, y_1) \geq F(x_1, y_2) - F(x_2, y_2). \tag{30}$$

If F is twice continuously differentiable, Topkis's (1978) Characterization Theorem says that supermodularity is equivalent to $\dfrac{\partial^2 F}{\partial x \partial y} \geq 0$, for all x, y. Furthermore, $\dfrac{\partial^2 F}{\partial x \partial y} > 0$ implies that F is strictly supermodular, the latter notion being defined by a strictly inequality in (30). Supermodularity is usually interpreted as a complementarity property: Having more of one variable increases the marginal returns to having more of the other variable.

F has the single-crossing property or SCP in $(x; y)$ if, for $x_1 \geq x_2$, $y_1 \geq y_2$,

$$F(x_1, y_1) - F(x_2, y_2) \geq 0 \Longrightarrow F(x_1, y_2) - F(x_2, y_1) \geq 0 \tag{31}$$

Note that (30) implies (31), while the converse is generally not true. Additionally, (30) is a cardinal notion while (31) is ordinal. Thus, the SCP is sometimes also referred to as ordinal supermodularity.

For $x \in R_+$, let $A(x) = [a_1(x), a_2(x)] \subset R_+$, with $a_1(.)$ and $a_2(.)$ being real-valued functions. $A(.)$ is ascending (in x) if a_1 and a_2 are increasing in x. The following results on monotone maximizers are central to our approach.

Theorem A.1. (Topkis (1978)). Assume that (i) F is upper-semi continuous (or u.s.c.) and supermodular in (x, y) and (ii) $A(.)$ is ascending. Then, the maximal and minimal selections of $y^*(x) \equiv \arg\max_{y \in A(x)} F(x, y)$ are increasing functions. Furthermore, if F is strictly supermodular, then every selection of $y^*(.)$ is increasing.

Theorem A.2. (Milgrom and Shannon (1994)). Assume that (i) F is u.s.c. and has the SCP in (x, y) and (ii) $A(.)$ is ascending. Then, the conclusion of Theorem A.1. holds.

The theorem that follows is a special case of Tarski's Fixed Point Theorem.

Theorem A.3. Let $n \geq 1$ and $B : X_{i=1}^n [a_i, b_i] \to X_{i=1}^n [a_i, b_i]$ be an increasing function. Then B has a fixed point.

Our equilibrium comparisons are based on the following result (Milgrom and Roberts,1990).

Theorem A.4. Let $B_t : X_{i=1}^{n}[a_i, b_i] \rightarrow X_{i=1}^{n}[a_i, b_i]$ be an increasing function, $\forall t$, such that $B_t(x)$ is also increasing in t, $\forall x$. Then the minimal and maximal fixed-points of B_t are increasing in t.

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